New Jersey Institute of Technology
Digital Commons @ NJIT

Theses | Theses and Dissertations

Summer 2007

Theoretical framework for predicting joint reaction and ground reaction forces in a dynamic pendulum tree model of human motion

Peyman Rajai
New Jersey Institute of Technology

Follow this and additional works at: https://digitalcommons.njit.edu/theses

Part of the Biomedical Engineering and Bioengineering Commons

Recommended Citation
Rajai, Peyman, "Theoretical framework for predicting joint reaction and ground reaction forces in a dynamic pendulum tree model of human motion" (2007). *Theses*. 396.
https://digitalcommons.njit.edu/theses/396

This Thesis is brought to you for free and open access by the Theses and Dissertations at Digital Commons @ NJIT. It has been accepted for inclusion in Theses by an authorized administrator of Digital Commons @ NJIT. For more information, please contact digitalcommons@njit.edu.

Copyright Warning & Restrictions

The copyright law of the United States (Title 17, United States Code) governs the making of photocopies or other reproductions of copyrighted material.

Under certain conditions specified in the law, libraries and archives are authorized to furnish a photocopy or other reproduction. One of these specified conditions is that the photocopy or reproduction is not to be "used for any purpose other than private study, scholarship, or research." If a, user makes a request for, or later uses, a photocopy or reproduction for purposes in excess of "fair use" that user may be liable for copyright infringement,

This institution reserves the right to refuse to accept a copying order if, in its judgment, fulfillment of the order would involve violation of copyright law.

Please Note: The author retains the copyright while the New Jersey Institute of Technology reserves the right to distribute this thesis or dissertation

Printing note: If you do not wish to print this page, then select "Pages from: first page # to: last page #" on the print dialog screen

The Van Houten library has removed some of the personal information and all signatures from the approval page and biographical sketches of theses and dissertations in order to protect the identity of NJIT graduates and faculty.

ABSTRACT

THEORETICAL FRAMEWORK FOR PREDICTING JOINT REACTION AND GROUND REACTION FORCES IN A DYNAMIC PENDULUM TREE MODEL OF HUMAN MOTION

by
Peyman Rajai

Lagrangian dynamics and the method of superfluous coordinates are applied to find ground and joint reaction forces on the human body modeled as a general branched 2-D pendulum tree system with arbitrary segments and arbitrarily distributed point masses. A theoretical framework is established for predicting these constraint forces during human motion and consequently their effects on dynamics, dynamic stability, energy efficiency and the potential of these forces to produce joint injury and/or pain. Applications to human walking are initiated. During idealized phases where there is only single point contact of the stance leg with the ground such as just after heel-strike and just before toe-off, the ground reaction force is modeled as the constraint force on the root pivot joint of the tree. Treating the length of the root segment as a superfluous coordinate introduces a new degree of freedom into the equations of motion that can be used to predict human movements that occur during flight such as in jumping, running and diving. The approach of adding an explicit constraint to the pendulum tree system is used at those times when the foot or a portion of it is flat on the ground. Proof of concept for this approach is demonstrated by application to a single pendulum constrained to lie horizontally on the ground and to a double pendulum system with the same constraint imposed on its first (root) segment.

THEORETICAL FRAMEWORK FOR PREDICTING JOINT REACTION AND GROUND REACTION FORCES IN A DYNAMIC PENDULUM TREE MODEL OF HUMAN MOTION

by
Peyman Rajai

A Thesis
Submitted to the Faculty of
New Jersey Institute of Technology
In Partial Fulfillment of the Requirements for the Degree of
Master of Science in Biomedical Engineering

Department of Biomedical Engineering

August 2007

Blank Page

APPROVAL PAGE

THEORETICAL FRAMEWORK FOR PREDICTING JOINT REACTION AND GROUND REACTION FORCES IN A DYNAMIC PENDULUM TREE MODEL OF HUMAN MOTION

Peyman Rajai

Dr. Michael H. Lacker, Thesis Advisor — Date
Professor of Biomedical Engineering, NJIT

Dr. William C. Hunter, Committee Member — Date
Professor of Biomedical Engineering, NJIT

Dr. Richard A. Foulds, Committee Member — Date
Associate Professor of Biomedical Engineering, NJIT

Dr. Sergei Adamovich, Committee Member — Date
Assistant Professor of Biomedical Engineering, NJIT

BIOGRAPHICAL SKETCH

Author: Peyman Rajai

Degree: Master of Science

Date: August 2007

Undergraduate and Graduate Education:

- Master of Science in Biomedical Engineering,
  New Jersey Institute of Technology, Newark, NJ, May 2007

- Bachelor of Science in Biomedical Engineering,
  New Jersey Institute of Technology, Newark, NJ, May 2005

Major: Biomedical Engineering

Presentations:

Peyman Rajai and Dr. Michael H. Lacker,

"Application of the Boundary Method to model and simulate mechanically stable and efficient gaits," The annual fall meeting of the BMES Society, Philadelphia, Pennsylvania, October 2004.

Dedicated to my parents

ACKNOWLEDGMENT

I would like to express my deepest appreciation and admiration to Dr. Michael Lacker, my mentor and my research advisor, who has guided me, helped me and inspired me since the beginning of my academic career at NJIT. He has been a constant instrument of motivation and drive in my progress and intellectual growth.

I have greatly enjoyed my educational experience in the Biomedical Engineering Department, having found myself surrounded by kind, caring and supportive faculty and staff members. Last but not least, I would like to thank Dr. William Hunter, Dr. Richard Foulds, and Dr. Sergei Adamovich for actively participating in my committee.

Special thanks are also given to Bob Narcessian, whose contagious enthusiasm and passion has been an immense source of inspiration.

TABLE OF CONTENTS

TABLE OF CONTENTS
(Continued)

CHAPTER 1

MOTIVATION

In the area of movement analysis, mathematical modeling seems to be the evolving method that "couples quantitative measures of human motion with theoretical concepts in physiology and mechanics" (Lacker, 1997). It is a design that investigates human movement in search of new motion. "These models are useful in better understanding movement and in estimating clinical parameters that are significant but otherwise difficult to access. For example, joint reaction forces and dynamic stability indices are clinically important but difficult to assess" (Lacker, 1997)

In modeling any movement task it is important to ideally select the most direct representation that can best describe the motion. This is usually easier said than done. The mechanical system model for any given motion is dynamic and the number of segments engaged in performing a task will vary in time. Therefore, the modeling itself is a dynamic process that requires a systematic approach capable of self-modification. The central inspiration for this thesis evolved around an attempt to formulate this modification process.

The dynamics of a mechanical system and the segments that are engaged during a motion will be restrained due to the existence of certain constraint forces. These forces may arise due to the presence of external constraints on a system such as the floor or they may be due to constraints on the skeletal system that restrain joint motion. With the progression of motion, the forces of constraint will generally change in time or may even disappear allowing additional degrees of freedom of movement to suddenly appear in the model.

Because it requires great skill to simultaneously coordinate and control many degrees of freedom it becomes necessary for the neuromuscular system to apply constraints on the human skeletal system to limit the number of moving segments during any phase of motion. The appropriate choice of which segments to move and what constraints to apply during the motion can often be non-intuitive since complex interactions occur when segments move together that give rise to new dynamic forces that can *only* be 'felt' when the system is in *motion* and that have significant effects both upon stability and mechanical efficiency.

Identification of these changes in the model requires continuous monitoring of constraint forces on those specific segments. The proficiency to assess the constraint forces at each joint will provide a fundamental method for modeling movement tasks that are more enhanced and that distinctively will be a better fit to the actual characteristics of a motion. Consequently, appropriate modeling methods present better insights for finding new motion techniques.

More significantly, the main objective of this thesis revolves around an attempt to compute the patterns of ground reaction forces during walking at different speeds. The ability to observe and monitor these forces prompts the prospect of identifying the prerequisites to model modification and the introduction or exclusion of model segments. In the walking example, these findings may also help in understanding both the transition from a walking gait to a running gait and the clarification of mechanical triggering mechanisms.

CHAPTER 2

BACKGROUND

2.1 Walking, Running and its Differences

There are two distinct definitions that describe the transition from a walking gait to a running gait. In *classical* terms 'walking' is a form of locomotion in which at least one leg is in contact with the ground at all times (Hildebrand, 1985) and where there are also usually relatively short phases of "double support" when both feet are in contact with the ground while in running there is a distinct phase where there no ground contact. In *biomechanical* terms however, the transition from a walking gait to a running gait is marked by a sudden and distinct change in the pattern of movement of the center of mass.

The distinct transition from one gait to the other can be observed in both the kinematics and kinetic patterns (Farley, 1998). There have been many studies examining a host of different kinematics variables as well as variables related to metabolic energy cost, but it is not yet clear exactly what triggers the transition from walking to running or vice-versa in humans or other animals (Farley, Getchell and Whitall). Margaria (1938) was among the first investigators to quantify that it is metabolically more expensive for humans to walk rather than to run at speeds faster than 2m/s, leading to the idea that there is a metabolic trigger for the walk–run transition. Conversely, Farley and Taylor (1991) showed that the trot–gallop transition speed in horses is not triggered metabolically but rather mechanically. Similarly, humans switch from a walk to a run at a speed that is not energetically optimal (Farley and Ferris, 1998). These findings suggest that other factors may actually trigger this gait transition.

2.2 Walking Models

The simplest model for walking is an inverted pendulum that idealizes the total body mass to a point mass on a rigid massless leg (Alexander, 1977). In this model, the gravitational potential energy of the mass is exactly 180° out of phase with the kinetic energy. This pattern of mechanical energy fluctuation is qualitatively similar to the pattern observed during walking in humans and other animals. In an idealized inverted pendulum model, 100% recovery of mechanical energy occurs due to the exchange between gravitational potential energy and kinetic energy. In humans, the pendulum-like mechanism conserves approximately 70 % of the mechanical energy from step to step at the preferred walking speed (approximately 1.3m/s) (Cavagna *et al.* 1976).

Based on the mechanics of an inverted pendulum system, it has been predicted that humans and other animals should be able to use a walking gait at speeds where the Froude number is less than or equal to 1 (Moretto, 1996). The Froude number (*Fr)* is a dimensionless ratio of inertial force to gravitational force. For legged locomotion, the inertial force typically used is the centripetal force acting on the animal as it arcs over a stance leg. Because body mass is in both the numerator and denominator, the Froude number reduces to:

$$Fr = (mv^2/L)/mg = v^2/gL \tag{2.1}$$

Where v is forward velocity, g is gravitational acceleration and L is the animal's leg length. Alexander and Jayes (1983) tested their hypothesis of dynamic similarity, by comparing the locomotion mechanics of small and very large animal species (from rodents to rhinoceroses) walking, running, trotting and galloping over a wide velocity range. They found that, despite very large differences in sizes and velocities, animals move with remarkably similar mechanics at equal values of the Froude number. In general, humans and other bipedal animals prefer to switch from a walk to a run at a Froude number of approximately 0.5 (Alexander, 1977, 1989; Gatesy and Biewener, 1991; Hreljac, 1995*b*; Thorstensson and Roberthson, 1987) which is substantially lower than the theoretically predicted Froude number.

Concisely, based on the inverted pendulum model, the transition from walking to running predominantly involves sudden changes in ground contact time, duty factor (the fraction of the stride time that a foot is in contact with the ground), movements of the center of mass and ground reaction force. In his "*Biomechanics of Walking and Running*", Farley states that "The distinct difference between walking and running gaits is apparent in the ground reaction force patterns for the two gaits". However, he further adds that "...although an inverted pendulum model does a good job of predicting the mechanical energy fluctuations of the center of mass, it does not accurately predict the ground reaction force pattern (1998)".

Mochon and McMahon (1980) developed a mathematical model of the swing phase of walking as a conservative coupled 3-segment pendulum system to predict the form of the swing period vs. walking speed relationship. Lacker and Choi (1997) later extended the coupled pendulum system model of Mochon and McMahon to include both swing and double support phases and added simple joint viscosity terms to account for both observed mechanical energy dissipation and the observed convexities in the segment angle curves of the shank and thigh segments of the swing leg as functions of time.

Although the idealization of a three-segment walking model appears sufficient for a 2 miles/hour walk, it may not represent the reality of a multi-segment walk for higher speeds of walking. In fact the predicted energy consumption curve in their studies showed that the discrepancy between the theoretical and experimental data became larger as the walking speed increased. In human walking, when the walking speed increases, the heel of the stance leg is lifted up before the heel of the swing leg contacts the ground. In order to expand the predictive utility of their model for higher walking speeds, Choi suggested the addition of a third walking phase between the start of swing phase and heel lift off of the stance leg which would also increase the number of segments of the model (Choi, 1997).

The ability to progressively revise a mechanical model by systematically introducing new phases where the number of dynamically interacting segments and model constraints can change may indeed be of foremost importance in understanding what mechanical triggers may be at play in motivating transitions between different patterns of movement. It is believed here that this approach may be of particular importance in gaining insights into the transition between walking and running. In fact even though a distinct transition from one gait to the other is evident in the kinematics, it could be possible that the bifurcation in qualitative locomotion technique may be linked to critical values of specific model parameters. For example, at critical speeds dynamic forces can suddenly appear, disappear or change sign and therefore result in the release of old constraints and/or the enforcement of new constraints in specific configurations of the moving mechanical system. This can prove to be a significant motivation for an individual or animal to transit to a new motion or gait pattern at a particular speed and body configuration.

It is the purpose of this thesis project to formulate the computational procedure that can monitor the ground and joint reaction force patterns which may trigger the introduction of additional segments into the system as walking speed increases. Consequently, iterative model modification will produce new dynamic models, afterward generating new sets of ground reaction force outputs. Ultimately, there may be a point at some walking speed or stride length or a combination of other model parameters, where the bifurcation between walking and running gait can be observed and mechanically understood, hence signifying an important mechanism triggering the gait change.

CHAPTER 3

METHOD

3.1 Introduction to Superfluous Coordinates: A Method for Finding Constraint Forces in Models of Lagrangian Dynamical Systems with Implicit Constraints

Generally in any dynamic mechanical system, forces of constraint can either be expressed in the dynamic EOM explicitly in their form as identified force vectors acting on any number of mass points or they may be implemented into the EOM implicitly, in which case, the constraints may be pre-imposed into the EOM in the form of constant parameters that satisfy the presence of those constraints. For example, consider a single frictionless pendulum moving in a constant vertically downward gravitational field of strength g, with massless rod of length l and point mass m at its end. The equation of motion for this simple system takes the form:

$$m\ddot{\phi} = -\frac{mg}{l}\sin\phi \tag{3.01}$$

where ϕ is defined as the counterclockwise angle that the pendulum rod makes with the downward ray of the y-axis. In this case, there is a force of constraint that is not explicit in the equation. To solve for the reactive force at the pivot, one needs to solve for the tension force in the rod which is acting as a constraint force, keeping the rod at a constant length and preventing the mass from taking off into the air.

One method that can be used to solve for constraint forces is to introduce superfluous coordinates into the system that in effect release the *implicit* constraints on the system. This change will alter the equations of motion of the system in such a way that the forces that are required to maintain the constraints are revealed when the constraints are *explicitly* re-applied to the altered equations.

In the example of the single pendulum above one considers the mass point of the pendulum as a free particle, hence introducing a superfluous coordinate that allows the length of the rod to be a new variable (r). Expressing the Lagrangian function (kinetic – potential energy) with the additional degree of freedom in this system and applying Lagrange's Method (see Section 3.2) results in different equations of motion of the following form:

$$\begin{cases} mr^2\ddot{\phi} = -mgr\sin\phi - 2mr\dot{r}\dot{\phi}, & (\hat{\phi}\ component) \\ m\ddot{r} = mg\cos\phi + mr\dot{\phi}^2, & (\hat{r}\ component) \end{cases} \tag{3.02}$$

Explicitly imposing the constraint condition $r = l$ (a constant) on the 1st equation results in reproducing the single pendulum Equation (3.02). Explicitly imposing the constraint condition $r = l$ (a constant) of the second Equation (3.02), reveals the constraint force:

$$F_c = -mg\cos\phi - ml\dot{\phi}^2 \tag{3.03}$$

This force is equal and opposite to forces acting on the free mass point in the $\hat{r}$ direction. These forces would lengthen or shorten the pendulum rod if it did not react to oppose them with equal and opposite force (tension) to maintain its constant length. In a single pendulum, $-mg\cos\phi$ comes from the static contribution of the constant gravitational force on the mass and $ml\dot{\phi}^2$ is the dynamic contribution due to the inertial centrifugal force.

Overall, in their implicit form the constraint forces are not readily quantifiable and in their explicit form, particularly for a complex mechanical system, they may not easily be identifiable. The application of superfluous coordinates in this context will prove to be invaluable especially when later applied to finding the joint reaction forces for a complex pendulum tree system.

3.2 Lagrange Equations of Motion without Explicit Constraints

The modeling approach for this thesis is based on a general pendulum tree system with a root base and branched segments that are distal to the root. The Lagrange method of dynamics will be applied for the derivation of the equations of motion (*EOM*) for the general pendulum tree. Lagrange's procedure is based on the scalar quantities of kinetic and potential energy of a conservative mechanical system. The Lagrangian, a scalar function of the independent dynamic variables of the system and their derivatives, is defined as:

(Lagrangian = Kinetic Energy - Potential Energy)

$$L = K(x,\dot{x}) - P(x), \tag{3.04}$$

where $x(t) = (x_1(t),\cdots,x_N(t))$ is the configuration of the system at time t expressed as a vector with N components (degrees of freedom) representing any (generalized) independent set of dynamic variables of the system and $\dot{x}(t) = v(t) = (\dot{x}_1(t),\cdots,\dot{x}_N(t))$ is the system's corresponding generalized velocity. For any conservative mechanical system with n degrees of freedom and without any explicit constraints on the system, the *EOM* of the system are obtained simply by applying the following analytic procedure for each component of the system:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \left(\frac{\partial L}{\partial x_i}\right) = 0, \qquad i = 1,\cdots,N. \tag{3.05}$$

For most mechanical systems the kinetic energy can be expressed as a quadratic form in the generalized coordinate system. More precisely:

$$K(x,\dot{x}) = \frac{1}{2}\dot{x}^T M(x)\,\dot{x} \tag{3.06}$$

This form is a generalized matrix vector form of the familiar kinetic energy for a single particle $K = \frac{1}{2}M\,v^2 = \frac{1}{2}\dot{x}^T M\,\dot{x}$. The matrix M in Equation (3.06) is both positive definite and symmetric but it is not in general a constant matrix but rather depends upon the system configuration x and therefore changes in time. It also need not have dimensions of mass. For the simple pendulum considered in the previous section (Section 3.1) with generalized coordinate $x \equiv \theta$,

$$K = \frac{1}{2}\dot{\theta}M\dot{\theta}, \quad M = ml^2 \text{ (Section 4.9)} \tag{3.07}$$

In the above equation we have defined θ as the counterclockwise angle that the pendulum rod makes with the positive x-axis. This is the standard definition of θ when using polar coordinates but it is different from the definition of ϕ that was used in Equation (3.01) for the simple pendulum where ϕ was defined as the counterclockwise angle that the pendulum rod makes with the negative y-axis. This is the usual choice made when considering small amplitude oscillations about the equilibrium point $\phi = 0$ where the approximation $\sin\phi \approx \phi$ is often made and Equation (3.01) reduces to the EOM of a simple harmonic oscillator. In this thesis, the standard polar coordinate definition of θ is used consistently when referring to the dynamic angle for any given pendulum segment, even in complicated multi-branched pendulum systems that can be used to model the human body (for example, see Figure 3.2).

When the kinetic energy can be expressed in the form of Equation (3.06) then the resulting system of differential equations in (3.05) can be expressed in a generalized matrix-vector form of Newton's second law (Appendix A):

$$M(x)\ddot{x} = F(x,\dot{x}) \tag{3.08}$$

For this reason and for the above analogy with the kinetic energy of a single particle the matrix M in Equations (3.06) thru (3.08) will be referred to as the generalized mass. Similarly F in Equation (3.08) will be referred to as the generalized force (even though it may not have units of force).

The simple pendulum Equation (3.01) expressed in the form of Equation (3.08) is:

$$\left(ml^2\right)\ddot{\theta} = F(x=\theta) = -mgl\cos\theta \qquad \text{(Section 4.9),} \tag{3.09}$$

where $\theta = \dfrac{\pi}{2} - \phi$ is the angle chosen to represent the dynamic variable as described in the paragraphs above. The generalized force is a torque that arises from the gradient of the potential energy $P(\theta) = mgl\sin\theta,\quad F(\theta) = -\dfrac{dP}{d\theta}$ and, in this case, has no $\dot{x} = \dot{\theta}$ dependence. The system (3.02) representing the motion of a free particle in polar coordinates would be represented in the form of Equation (3.08) by:

$$\begin{pmatrix} mr^2 & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{r} \end{pmatrix} = F\left(x = \begin{pmatrix} \theta \\ r \end{pmatrix}, \dot{x} = \begin{pmatrix} \dot{\theta} \\ \dot{r} \end{pmatrix}\right) = \begin{pmatrix} -mgr\cos\theta \\ -mg\sin\theta \end{pmatrix} + \begin{pmatrix} -2mr\dot{r}\dot{\theta} \\ mr\dot{\theta}^2 \end{pmatrix} \tag{3.10}$$

In this case the generalized force is the sum of two vectors; the first arises from the gradient of the potential energy, $P(x = \begin{pmatrix} \theta \\ r \end{pmatrix}) = mgr\sin\theta,$

$$F_{grav}(x) = -\nabla P(x) = -\begin{pmatrix} \dfrac{\partial P}{\partial \theta} \\ \dfrac{\partial P}{\partial r} \end{pmatrix}, \tag{3.11}$$

and the second term arises from derivatives of the kinetic energy,

$$K(x,\dot{x}) = \frac{1}{2}\dot{x}^T M(x)\,\dot{x} = \frac{1}{2}\begin{pmatrix}\dot{\theta} & \dot{r}\end{pmatrix}\begin{pmatrix} mr^2 & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix}\dot{\theta} \\ \dot{r}\end{pmatrix} = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2, \tag{3.12}$$

and therefore represent inertial forces. The θ-component inertial force ($-2mr\dot{r}\dot{\theta}$) is the Coriolis force and r-component inertial force ($mr\dot{\theta}^2$) is the centrifugal force. The θ-component generalized force is a torque while the r-component generalized force has units of force.

3.3 Lagrange Equations of Motion for a General Branched Coupled 2-D Pendulum Tree without Explicit Constraints

Consider a branched pendulum tree system in 2-D, consisting of S segments with P distributed point masses on the segments (Figure 3.1).

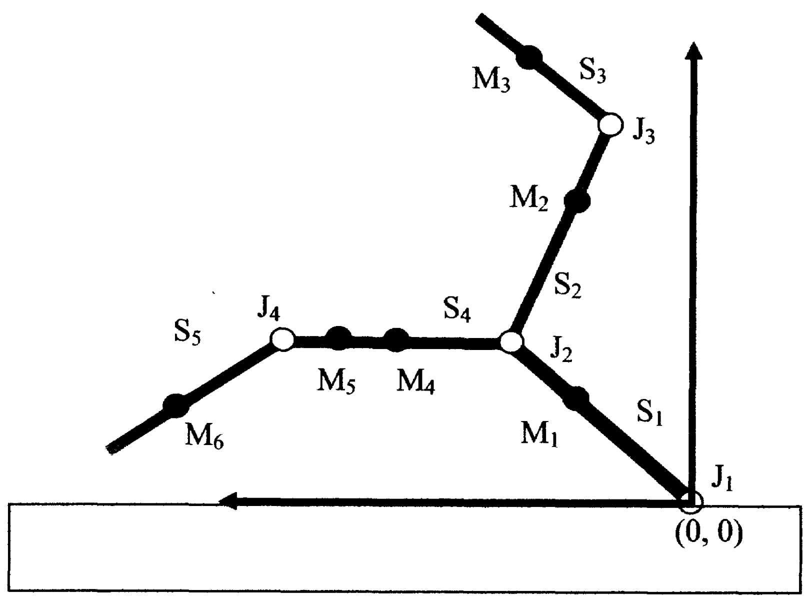

Figure 3.1 A 2-D pendulum tree with six point masses and five segments and four joints.

Let the point that attaches the pendulum tree to the lab frame (ceiling, wall or floor) be the origin of a coordinate system and let the root segment that attaches to the lab at the origin be denoted as segment 1. All other segments of the tree will be connected or descended from this root segment. Each segment of the tree belongs to a generation of the tree depending upon how far removed it is from the root segment. If it is directly connected to the root, it is a generation 1 or child of the root. If it is directly connected to a child of the root, then it is a generation 2 or grandchild of the root and so on.

Label the segments with numbers in order of their relation to the root in such a way that each child of a segment has a number higher than any of its parent's sibling segments (aunts or uncles). Every branch point of this tree will represent a joint of the pendulum system and all joints except the origin of the system will have at least one proximal and one distal joint segment attached to it. The origin pivot joint will have only the root segment attached to it and will be labeled the root joint and given the number 1.

All joints will be numbered so that every joint more distally related to the root joint will have a higher joint number than a more proximally related joint. Number the mass points (the fruit) on the tree in a similar fashion, i.e., so that no mass point on a given segment has a lower number than another mass point that is on a segment that is more central to the given segment, and so that all mass points that are on the same segment are numbered in order of their distance from the most proximal joint of that segment.

Let the length of the j^{th} segment be denoted by L_j and let the distance of the ith mass point from its most proximal joint be denoted by z_i. Define a matrix R called the relation matrix whose entries consist of 0's and the lengths z_i and L_j. There are S columns in R and P rows. The i^{th} row refers to the i^{th} mass point and the j^{th} column to j^{th} segment. If the j^{th} segment has the i^{th} mass point on it then the entry $R_{i,j} = z_i$. If the j^{th} segment is a parent, grandparent, great-grandparent, or any fore-parent of the segment that the i^{th} mass point is on then $R_{i,j} = l_i$. For all other entries $R_{i,j} = 0$. The relation matrix for Fig. 3.2 is therefore given by:

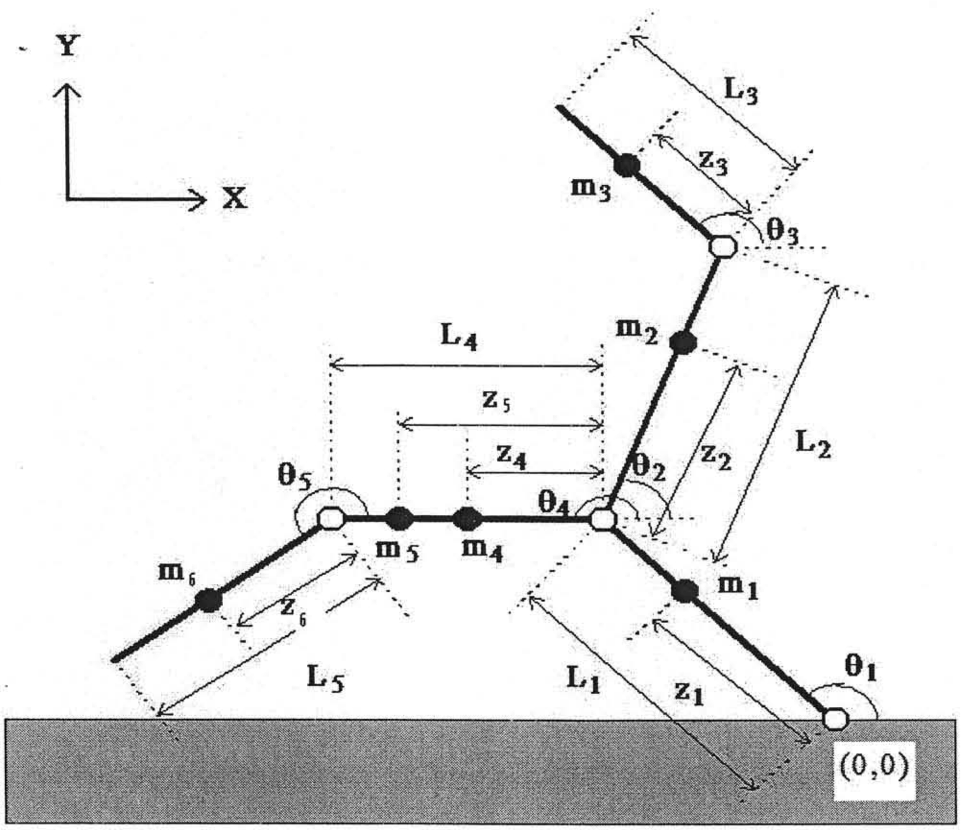

Figure 3.2 The same dynamic pendulum tree as in Figure 3.1 but with the dynamic angles labeled as well as the segment lengths and distances of the centers of mass from their proximal joints.

$$\mathbf{R} = \begin{bmatrix} z_1 & 0 & 0 & 0 & 0 \\ L_1 & z_2 & 0 & 0 & 0 \\ L_1 & L_2 & z_3 & 0 & 0 \\ L_1 & 0 & 0 & z_4 & 0 \\ L_1 & 0 & 0 & z_5 & 0 \\ L_1 & 0 & 0 & L_4 & z_6 \end{bmatrix} \tag{3.13}$$

Let $\theta_i(t)$ be the dynamic angle (counterclockwise positive) that the ith segment makes with the positive ray parallel to the x-axis that passes through its proximal joint (Figure 3.3.2), then a generalized configuration, x, for the pendulum tree can be defined as an S-vector $x = \theta = (\theta_1, \cdots, \theta_S)^T$. For the specific pendulum system represented by (Figure 3.2), there are 6 segments and therefore S=6 (degrees of freedom) in the dynamic system. (Choi, 1997) has shown that the kinetic energy for the generalized 2-D pendulum tree can be expressed as:

$$\begin{aligned} &K = \frac{1}{2}\sum_{i=1}^{P} m_i v_i^2 = \frac{1}{2}\dot{\theta}^T M(\theta)\,\dot{\theta} \\ &M_{i,j} = C_{i,j}\cos(\theta_i - \theta_j) \qquad , \\ &C = R^T M_D R \end{aligned} \tag{3.14}$$

where,

$$\mathbf{M}_D = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & m_P \end{bmatrix} \tag{3.15}$$

In this case the mass matrix M is positive definite and symmetric but not a constant matrix. It changes with system configuration and therefore is dynamic. The potential energy is

$$P = \sum_{i=1}^{P} m_i\, g\, y_i = g \sum_{j=1}^{S} \left(\underline{M}^T R \right)_j \sin\left(\theta_j\right), \tag{3.16}$$

where $\underline{M}^T = \left(m_1, m_2, \cdots, m_P\right)$. Lagrange EOM for the coupled 2-D dynamic pendulum tree system expressed in the generalized form of Newton's second Law is:

$$M\ddot{\theta} = -\nabla P + S(\theta)\,\dot{\theta}^2, \tag{3.17}$$

where,

$$\left(\nabla P\right)_i = \frac{\partial P}{\partial \theta_i} = g\left(\underline{M}^T R\right)_i \cos\left(\theta_i\right), \tag{3.18}$$

and $S\dot{\theta}^2$ is a generalized inertial force vector obtained by multiplying a skew-symmetric matrix whose components are defined by $S_{i,j} = C_{i,j}\sin(\theta_j - \theta_i)$ by the vector $\dot{\theta}^2 = \left(\dot{\theta}_1^{\,2}, \dot{\theta}_2^{\,2} \cdots, \dot{\theta}_S^{\,2}\right)^T$ (Appendix A). If there are joint dissipative viscous forces $f_j = -b_j\dot{\alpha}_j$ that oppose the change in the joint angle α formed by any joining 2 segments and that are proportional to the joint's angular velocity $\dot{\alpha}_j$ with viscous joint coefficient b_j then a third term will appear in the generalized force on the right hand side of Equation (3.17). This term will take the form $B\dot{\theta}$ where the matrix B is a constant matrix depending only on the joint coefficients and the EOM for the system will take the form:

$$M\ddot{\theta} = -\nabla P + S(\theta)\,\dot{\theta}^2 + B\dot{\theta} \tag{3.19}$$

It should be noted that when the generalized coordinates are angles as is the case here with the segment angles, then the equations of motion that result from applying Lagrange's method to those components of the system will correspond to Newton's 2nd Law for rotating systems. More precisely, the generalized mass terms will be moments of inertia, *I*, associated with the rotating mass points, the generalized forces will be expressed as torques or, Γ, acting through a moment arm about the axis of rotation of a joint associated with the rotating segment, and the angular accelerations will relate these two quantities through the usual concept that the rate of change of angular momentum $\dot{L} = I\dot{\omega} = I\ddot{\theta} = \Gamma$. Torques arise on this pendulum tree system when the masses are acted upon by the external gravitational field and in human motion significant torques would also arise due to action of external muscle forces acting on the joints of the skeletal pendulum tree. Although all the mass points lie on the axis of a segment in this pendulum tree system, masses that attach to a given segment off the axis can be included in the general framework by making an additional branch off that segment and applying suitable constraints so as to keep the mass in a fixed position relative to it (Section 3.6).

3.4 Solving Lagrange Equations of Motion without Explicit Constraints for Initial Value Problems (IVP)

Equation (3.08) is not only a generalization of Newton's second law but more importantly it is a useful form for numerical solution of the EOM. More specifically, if x and $\dot{x}$ are known at time t then so are M and F. Therefore, Equation (3.08) can then be viewed as a standard $Ay = b$ linear algebra problem whose solution finds $y = \ddot{x}$ at time t. The known values of both $\dot{x}$ and $\ddot{x}$ at time t can then be used to estimate the values of x and $\dot{x}$ at the next numerical time step $t + \Delta t$, where Δt is a numerical parameter that is chosen to be sufficiently small so that numerical accuracy and stability can be achieved.

In the simplest estimation technique (explicit Euler Method) which is first order accurate in Δt, the values of x and $\dot{x}$ at the next numerical time step $t + \Delta t$ are given by;

$$\begin{cases} x(t+\Delta t) \\ \dot{x}(t+\Delta t) \end{cases} = \begin{matrix} x(t)+\Delta t\,\dot{x}(t) \\ \dot{x}(t)+\Delta t\,\ddot{x}(t) \end{matrix} \tag{3.20}$$

Now that if x and $\dot{x}$ are known at time $t + \Delta t$ the procedure described above of using Equation (3.08) to solve the linear algebra problem this time for $y = \ddot{x}$ at time $t + \Delta t$ can be repeated with repeated application of Equation (3.20) to obtain x and $\dot{x}$ at the next numerical time step $t + 2\,\Delta t$. Applying this procedure iteratively gives the numerical solution of the EOM at successive time steps,

$$\begin{cases} x(i\Delta t) \\ \dot{x}(i\Delta t), \quad i = 0,1,2,3,\cdots \\ \ddot{x}(i\Delta t) \end{cases} \tag{3.21}$$

where the *initial values* of x and $\dot{x}$ *at* $t=0$, $(i=0)$ *are given to start the process.* The numerical solutions often employ an estimation technique that is 4th order accurate in Δt (explicit 4th order Runga-Kutta) with an algorithm that adjusts Δt at each time step to achieve a preset accuracy (Numerical Recipes, 1992).

3.5 Geometric Interpretation of Explicit Constraints and the Generalized Forces of Constraint

Consider a system with *N* degrees of freedom and with K explicit constraints of the form;

$$G_i(x) = 0, \qquad i = 1,\cdots,K, \quad (K \leq N) \tag{3.22}$$

Geometrically in configuration space each constraint reduces the available space for the solution by *1*-Dimension and therefore the solution must lie on a *(N-1)* dimensional surface in configuration space that satisfies that constraint equation. Satisfying *K* constraints implies that the solution trajectory *x(t)* must lie on an *(N-K)* dimensional surface that represents the intersection of the *K* and *(N-1)* dimensional surfaces. At any point *P* on this constraint surface the tangent "plane" is an *N-K* dimensional hyper-plane (subspace with P as origin). Any velocity vector in configuration space that is drawn from P of a solution trajectory that satisfies the constraints and passes through P must lie in this *N-K* dimensional hyper-plane.

Since $G_i(x) = 0$ for any trajectory x*(t)* on the constraint surface then by the chain rule the derivative $\left.\frac{dG_i(x(t))}{dt}\right|_P$ must satisfy:

$$0 = \left.\frac{dG_i(x(t))}{dt}\right|_P = \nabla G_i^T(x(t))\Big|_P \left.\frac{dx(t)}{dt}\right|_P = \nabla G_i^T\Big|_P \dot{x}\Big|_P \tag{3.23}$$

Equation (3.23) implies that the gradient vector of the ith constraint must be orthogonal to the tangent space at P of the constraint surface. Let $\nabla G_P = (\nabla G_1, \nabla G_2, \ldots \nabla G_k)_P$ define the $N \times K$ matrix whose column vectors are orthogonal to the constraint surface at *P*.

The generalized force of constraint $F_C(x(t))\big|_P$ *must be acting at P in a direction that is orthogonal to the constraint surface to provide the appropriate* <u>*reactive*</u> *force at P that will prevent the trajectory from moving off the surface.* Mathematically this implies that $F_C(x(t))\big|_P$ must be some vector that is a linear combination of the columns of ∇G_P that form a basis for the orthogonal subspace to the constraint surface at *P*. More precisely,

$$F_C(x(t))\Big|_P = -\sum_{i=1}^{K} \nabla G_i(x(t)\, \lambda_i(t)\Big|_P = -\nabla G(x(t))\, \lambda(t) \tag{3.24}$$

The minus sign is conventional to indicate that the constraint force is acting equal to, but in a direction that is *opposite* to, that component of the generalized force that is orthogonal to the constraint surface and that would be present at *P* if *no constraints were acting on the system.* The vector function $\lambda(t) = (\lambda_1(t), \cdots, \lambda_K(t))^T$ is called the Lagrange Multiplier. To find, $F_C(t)$, the generalized constraint force at time *t*, given the configuration of the system at time $x(t)$, it is only necessary to find $\lambda(t)$ since $\nabla G(x)$ is known from the given explicit constraints.

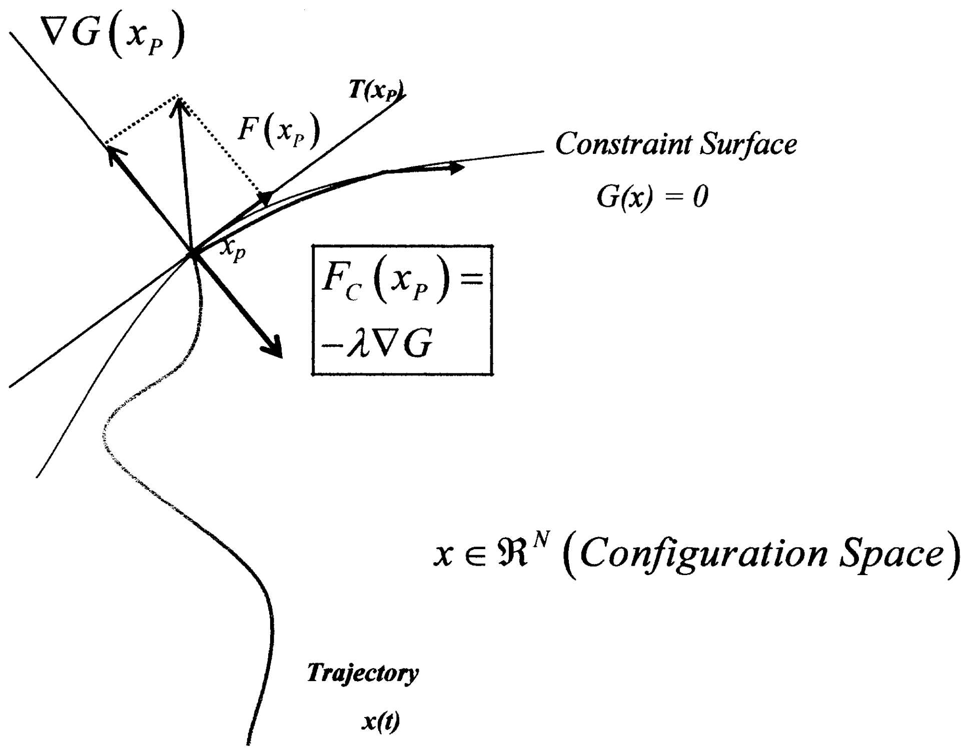

Figure 3.3 Geometric interpretation of the generalized force of constraint in configuration space.

3.6 Lagrange Equations of Motion with Explicit Constraints

The arguments of the previous section, demonstrate that the EOM for a system with N degrees of freedom and K constraints $G_i(x) = 0, \quad i = 1, \cdots, K, \quad (K \leq N)$ will take the form of Equation (3.08) with an additional term on the right hand side that represents the generalized constraint force, $F_C = -\nabla G\, \lambda(t)$,(Lanczos, 1970;McCauley,1997) that is, :

$$M(x)\ddot{x} = F(x,\dot{x}) - \nabla G\, \lambda(t) \tag{3.25}$$

This system, however, has only N equations and $N+K$ unknowns. Since both $\ddot{x}(t)$ and $\lambda(t)$ are unknowns in an IVP (section 3.4). The additional K equations needed to complete the system come from the K explicit constraints but these equations are of a different type since they are not second order differential equations in t. They are converted to 'acceleration-like' equations by differentiating Equation (3.23) in time. If the result is appended to Equation (3.25), then an extended form of Equation (3.08) (Newton's 2nd Law) $M_e(x)\ddot{x}_e = F_e(x,\dot{x})$ results that in block matrix form is given by:

$$\begin{pmatrix} M(x) & \nabla G(x) \\ \nabla G^T(x) & 0 \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} F(x,\dot{x}) \\ -h(x,\dot{x}) \end{pmatrix} \qquad (3.26)$$

(Choi, 1997; Schenk 2000)

The first block row of Equation (3.26) is simply the same as Equation (3.25). The extended generalized mass matrix $M_e(x)$ has the generalized mass matrix M embedded in it as well as the constraint gradient matrix to fill out a symmetric $(N+K) \times (N+K)$ matrix that just depends upon the configuration x as was the case for Equation (3.08). The generalized extended acceleration vector contains the lambda Lagrange multiplier vector appended to the generalized acceleration vector and the extended generalized force vector contains an additional h vector appended to it whose K components satisfy,

$$h_i = \dot{x}^T H^{(i)}(x)\dot{x} \quad i = 1,\cdots,K \ , \qquad (3.27)$$

where the components of each of the K-symmetric $(N \times N)$ matrices $H^{(i)}$ $i = 1,\cdots,K$ are given by:

$$H^i_{j,l}(x) = \frac{\partial^2 G_i}{\partial x_j \partial x_l}, \quad j,l = 1,\cdots,N; \ i = 1,\cdots,K \le N \qquad (3.28)$$

3.7 Solution of both the Lagrange Equations of Motion with Explicit Constraints for Initial Value Problems and the Dynamic Generalized Constraint Forces

Since the form of the EOM with constraints, Equation (3.26), is identical to that of Equation (3.08) without constraints, therefore the method of solution described in Section 3.4 applies. When the linear algebra problem is solved for the extended system of Equation (3.26) the solution vector y contains at each time step both the acceleration of the constrained system $\ddot{x}(t)$ and the Lagrange multiplier $\lambda(t)$. The solution to the EOM for the constrained system will therefore consist of:

$$\begin{cases} x(i\Delta t) \\ \dot{x}(i\Delta t) \\ \ddot{x}(i\Delta t) \\ \lambda(i\Delta t) \end{cases}, \quad i = 0,1,2,3,\cdots \tag{3.29}$$

where the *initial values* of x and $\dot{x}$ *at* $t=0$, ($i=0$) *are given to start the process.*

The solution of the generalized force of constraint $F_C(i\Delta t)$ can be appended to Equation (3.29) since it can be computed at each time step by the matrix vector multiplication:

$$F_C(i\Delta t) = -\nabla G\big(x(i\Delta t)\big)\,\lambda(i\Delta t), \; i = 0,1,2,3,\cdots \tag{3.30}$$

Then, for completeness, at each time step the numerical solution will consist of:

$$\begin{cases} x(i\Delta t) \\ \dot{x}(i\Delta t) \\ \ddot{x}(i\Delta t) \\ \lambda(i\Delta t) \\ F_C(i\Delta t) \end{cases}, \quad i = 0,1,2,3,\cdots \tag{3.31}$$

When the constraint is active and the numerical computation starts it is important that the initial velocity vector lie in the tangent space of the constraint surface, that is, it must satisfy $\nabla G(x(0))\,\dot{x}(0) = 0$. If the solution trajectory just before the constraint applies hits the constraint surface at a point P then in general the trajectory will not at this time have a velocity in the tangent plane of the constraint surface but the forces of constraint (impact force) at the next instant of time will enforce the constraint and therefore a sudden change in the system velocity will occur to appropriately project the velocity onto the tangent plane at P to satisfy the constraint.

The physically correct projection calculation conserves as much of the generalized momentum of the system before the (inelastic) collision as possible consistent with satisfying the constraint after the collision (Hatze, 1981; Schenk, 2000). In general some of the momentum and energy of the system is lost at this time unless the system hits the constraint surface with a generalized velocity that is already tangent to the constraint surface at P. This has implications in finding human motion techniques that minimize energy and momentum losses (Choi, 1997; Schenk 2000).

CHAPTER 4

APPLICATION

4.1 Calculating the Dynamic Generalized Constraint Forces Associated with Implicit Constraints using the Method of Superfluous Coordinates

The method of superfluous coordinates releases *implicit* constraint(s) in the dynamical system by introducing new dynamic coordinate(s) into the system. The EOM are rewritten in terms of all the dynamic variables including the superfluous coordinates to obtain a form consistent with Equation (3.08) for the new unconstrained system. The constraints are then *explicitly* reintroduced into the system resulting in an extended system of the form of Equation (3.26). This system is then solved using the methods described in Section 3.7 and 3.4 to obtain both the generalized constraint forces and the motion of the constrained system. This technique will be applied in the next sections to obtain the generalized pivot reaction force *(PRF)* on the root joint of a branched pendulum tree whose equations were developed in Section 3.3.

4.2 Generalized EOM for a 2-D Dynamic Pendulum Tree with Superfluous Coordinate for the Position of the First Mass Center on the Root Segment

Superfluous coordinate: $z_1(t)$ (Figure 3.2)

Dynamic variables:

$$x(t) = \left(z_1(t), \theta(t)\right), \quad \theta(t) = \left(\theta_1(t), \ldots, \theta_N(t)\right), \quad (N+1)\ DOF\ \text{system} \qquad \text{(Figure 3.2)}$$

Constraint: $z_1 = C$ (a constant) or,

$$G(x) = z_1 - C = 0 \tag{4.01}$$

$$\nabla_i G(x) = \frac{\partial G}{\partial x_i} = \begin{cases} 1, & i = 1 \\ 0, & i \neq 1 \end{cases} \tag{4.02}$$

Let the *Cartesian* coordinates of the i^{th} mass point be denoted by $\left(\hat{x}_i, \hat{y}_i\right)$:

$$\hat{x}_i = \begin{cases} z_1 \cos\theta_1 , & i = 1 \\ l_1 \cos\theta_1 + \sum_{j=2}^{S} R_{ij} \cos\theta_j , & i = 2,3,\cdots,S \end{cases} \tag{4.03}$$

where R is the relation matrix defined in Section 3.3 for a generalized branched pendulum tree with P mass points and S segments. Similarly, for the Cartesian y-coordinate of the i^{th} mass point:

$$\hat{y}_i = \begin{cases} z_1 \sin\theta_1 , & i = 1 \\ l_1 \sin\theta_1 + \sum_{j=2}^{S} R_{ij} \sin\theta_j , & i = 2,3,\cdots,S \end{cases} \tag{4.04}$$

The Cartesian velocity components of the i^{th} mass point is given by:

$$\dot{\hat{x}}_i = \dot{z}_1 \cos\theta_1 - \sum_{j=1}^{S} R_{ij} \sin\theta_j \, \dot{\theta}_j \tag{4.05}$$

$$\dot{\hat{y}}_i = \dot{z}_1 \sin\theta_1 + \sum_{j=1}^{S} R_{ij} \cos\theta_j \, \dot{\theta}_j \tag{4.06}$$

Therefore the velocity squared of the i^{th} mass point is:

$$v_i^2 = \dot{x}_i^2 + \dot{y}_i^2 = (v_i^2)_o + \dot{z}_1^2 + 2\dot{z}_1 \sum_{j=1}^{S} R_{ij} \sin\left(\theta_1 - \theta_j\right) \dot{\theta}_j , \tag{4.07}$$

where, $$(v_i^2)_o = \left(\sum_{j=1}^{S} R_{ij} \cos\theta_j \, \dot{\theta}_j \right)^2 + \left(\sum_{j=1}^{S} -R_{ij} \sin\left(\theta_j\right) \dot{\theta}_j \right)^2 \tag{4.08}$$

represents the velocity squared of the i^{th} mass point in the dynamic pendulum tree system without the superfluous coordinate $z_1(t)$.

The kinetic energy of the system with superfluous coordinate is therefore given by:

$$K = \frac{1}{2}\sum_{i=1}^{P} m_i v_i^2 = K_o + \frac{1}{2} m_T \dot{z}_1^2 + \dot{z}_1 u^T \dot{\theta} \tag{4.09}$$

where K_o is the kinetic energy of the system without the superfluous coordinate given by Equation (3.14), and

$$m_T = \sum_{i=1}^{P} m_i \tag{4.10}$$

is the total mass of the system. The vector $\boldsymbol{u}$ is defined by:

$$u_j = \left(\underline{M}^T R\right)_j \sin(\theta_1 - \theta_j), \quad j = 1, \ldots, S \tag{4.11}$$

where ,

$$\left(\underline{M}^T R\right)_j = \sum_{i=1}^{P} m_i R_{ij}, \quad j = 1, \cdots, S \tag{4.12}$$

The Potential Energy for the system with superfluous coordinates is the same as that for the system without superfluous coordinates except that z_1 is now viewed as a dynamic variable. The component of the gradient of potential energy with respect to z_1 yields the generalized gravitational force in the $\hat{z}_1$ direction:

$$F_{grav}\Big|_{z_1} = -\frac{\partial P}{\partial z_1} = -m_T g \sin\theta_1 \tag{4.13}$$

All other components (in the $\hat{\theta}_i$ directions) of the generalized gravitational force F_{grav} are obtained by computing,

$$F_{grav}\Big|_{\theta_i} = -\frac{\partial P}{\partial \theta_i}, \tag{4.14}$$

which yields the same result as that given by Equation (3.18).

The Lagrangian function $L(x,\dot{x}) = K(x,\dot{x}) - P(x)$, where $x = (z_1, \theta)$ is obtained from the kinetic and potential energies developed in the above paragraphs for the general pendulum tree system with superfluous coordinate z_1. Systematic application of Lagrange's Equation (3.05) for each component (degree of freedom) yields the following generalized form of Newton's second Law with *S+1* independent dynamic variables and *S+1* equations (see Appendix B):

$$\begin{pmatrix} m_T & u^T \\ u & M \end{pmatrix} \begin{pmatrix} \ddot{z}_1 \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} w^T \dot{\theta}^2 - g m_T \sin(\theta_1) \\ S\dot{\theta}^2 - \nabla P + B\dot{\theta} + Cor \end{pmatrix}, \tag{4.15}$$

where m_T and u are defined as in Equations(4.10) and (4.11). The symbols for *M, S, B*, ∇P, $\dot{\theta}, \ddot{\theta}$ and $\dot{\theta}^2$ are defined as in Equation (3.19) for the *N*-degree of freedom system that defined the generalized pendulum tree system without superfluous coordinate z_1 and with viscous joint dissipation. The vector w has components defined by:

$$w_i = \left(\underline{M}^T R\right)_i \cos(\theta_1 - \theta_i), \quad i = 1, \cdots, S \tag{4.16}$$

and the vector *Cor* is the generalized Coriolis force with components,

$$Cor_i = -2\dot{z}_1 \dot{\theta}_1 w_i \qquad \text{(Appendix B)} \tag{4.17}$$

4.3 Generalized EOM for a Pendulum Tree Model of Human Dynamics in Flight

In the previous section the variable z_1 was viewed as a superfluous coordinate that frees the constraint for the pendulum tree to be rooted to the ground. This is considered as a first step in obtaining the reaction force on the root pivot (*PRF*). If in fact, the point of view is adopted that the superfluous coordinate is a ***real*** dynamic variable then the system represented by Equation (4.15) represents the equations that could model the human as a branched pendulum system in flight, that would apply for example, to human jumping and diving as well as that phase of human running in which both legs are off the ground. For the derivation of Equation (4.15) see Appendix B.

4.4 Generalized EOM for Extended 2-D Dynamic Pendulum Tree with Superfluous Coordinate and Explicit Constraint

To complete the procedure for finding the Pivot Reaction Force (*PRF*) on the root joint it is necessary to explicitly re-introduce the constraint on the length of the first segment of the pendulum into Equation (4.15). This can be accomplished by forming the extended system represented by Equation (3.26), with ∇G given by Equation (4.02). More precisely, the extended generalized form of Newton's second Law (with $N+2$ independent dynamic variables and $N+2$ equations) takes the form:

$$\begin{pmatrix} m_T & u^T & 1 \\ u & M & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \ddot{z}_1 \\ \ddot{\theta} \\ \lambda \end{pmatrix} = \begin{pmatrix} w^T\dot{\theta}^2 - gm_T\sin(\theta_1) \\ S\dot{\theta}^2 - \nabla P + B\dot{\theta} + Cor \\ 0 \end{pmatrix} \qquad (4.18)$$

The (0) in the last row of the generalized force vector on the right hand side of Equation (4.18), arises from applying Equation (3.27) and (3.28) to Equation (4.01) which in this case is zero:

$$\frac{d^2G}{dt^2} = \ddot{z}_1 = 0$$

4.5 The Dynamic Reaction Force at the Root Pivot

The first component of the system (4.18) is in the $\hat{z}_1$ direction and it is the only component of the system that contains the Lagrange multiplier λ. This implies that the constraint force is entirely in this direction. This is physically correct since the tension in the root segment is in this direction. From Equation (4.18) the EOM in this direction is:

$$\begin{pmatrix} m_T & u^T & 1 \end{pmatrix} \begin{pmatrix} \ddot{z}_1 \\ \ddot{\theta} \\ \lambda \end{pmatrix} = w^T \dot{\theta}^2 - g m_T \sin(\theta_1) \tag{4.19}$$

Solving the above equation for $-\lambda$ gives the force of constraint acting on the pivot or the Pivot Reaction Force (*PRF*) as:

$$\boxed{\begin{aligned} & -\lambda \hat{z}_1 = \left(u^T \ddot{\theta} - w^T \dot{\theta}^2 + m_T g \sin\theta_1\right) \hat{z}_1 \\ & where, \\ & m_T = \sum_{i=1}^{P} m_i \\ & \underline{m}^T = (m_1, m_2, \ldots, m_P) \\ & u_i = \left(\underline{m}^T R\right)_i \sin(\theta_1 - \theta_i), \quad i = 1, \cdots, S \\ & w_i = \left(\underline{m}^T R\right)_i \cos(\theta_1 - \theta_i), \quad i = 1, \cdots, S \end{aligned}} \tag{4.20}$$

R is the *(P × S)* Relation Matrix of segment lengths and mass centers defining an inheritance relationship between the P point masses and S segments of any given branching pendulum tree as defined in Section 3.3. Since z_1 has units of length, the generalized constraint force has units of force (the tension along the root segment).

4.6 Calculating the Dynamic Reaction Force at the Root Pivot

Formula (4.20) can be used to calculate the *PRF* only if the vector trajectory $\theta(t)$ and its derivatives $\dot{\theta}(t)$ and $\ddot{\theta}(t)$ of the pendulum system are known. Two approaches can be used to calculate the Root Pivot Reaction Force. One approach is to solve for $\theta(t)$, $\dot{\theta}(t)$, $\ddot{\theta}(t)$, $\lambda(t)$, and $-\nabla G(\theta(t))\lambda(t)$ simultaneously for each time step using the extended system (4.18) and the methods described in Section 3.7.

A second approach would take advantage of the fact that when the constraints are active, the superfluous coordinates are not dynamic variables. More precisely, since $z_1 = C \Rightarrow \dot{z}_1 = \ddot{z}_1 = 0$, therefore, the Coriolis term $Cor = 0$ in Equation (4.15) and the dynamics of the constrained system are given by the reduced system with implicit constraints (Equation (3.17) or (3.19)). Since this system has fewer degrees of freedom than Equation (4.18), it is more efficient to solve for $\theta(t)$, $\dot{\theta}(t)$, $\ddot{\theta}(t)$ by the method described in Section (3.4). Once these vectors are known at any time *t* the reaction force at the root pivot point can be calculated at that time by substituting directly into Equation (4.20). It should also be noted that Equation (4.20) could be applied to kinematical data collected in motion analysis laboratories to calculate the *PRF* provided that the data collected can be manipulated to reliably yield the segment angle vector trajectory $\theta(t)$ and its derivatives $\dot{\theta}(t)$ and $\ddot{\theta}(t)$ as well as appropriate structural parameters such as segment lengths, masses, and distribution of mass.

4.7 Generalized Joint Reaction Forces

Consider the joint J_i formed in the pendulum tree at the intersection of two segments S_{i-1} and S_i (Figure 3.1). Let S_i be the more distal segment to the joint and let S_{i-1} be the more proximal segment (relative to the root). A simple double pendulum with one 'joint' besides the root pivot is shown in Figure (4.1).

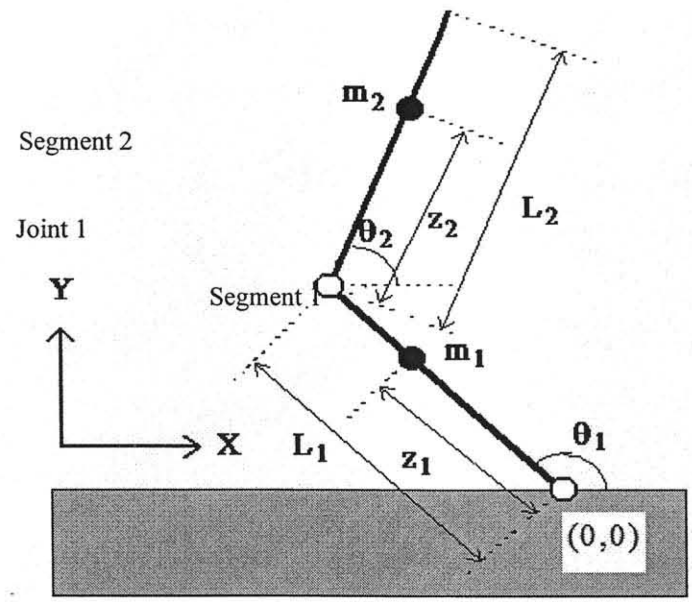

Figure 4.1 A simple double pendulum with root pivot and joint 1 at the intersection of segment 1 and segment 2.

The reaction force at any joint of the pendulum tree system is the vector sum of the reaction forces (tensions) acting at each segment forming the joint. Defining a superfluous coordinate for each segment feeding into or out of the joint and applying the same methods as described in Sections 4.2-4.7 for the reaction force at the root pivot will yield the dynamic tension acting on each joint segment. Each segment will have its Lagrange Multiplier that scales the unit vector acting along the segment to yield the tension on that joint segment.

As an example, this idea will be applied to calculate the joint reaction force at joint 1 of the simple 2-pendulum system shown in Figure 4.1. By defining two superfluous dynamic coordinates $z_1(t)$ and $z_2(t)$ a 4-degree of freedom dynamical system is produced with generalized coordinates $x=(z,\theta)$ where $z=(z_1,z_2)$ and $\theta=(\theta_1,\theta_2)$. The explicit constraints that produce constant joint segment lengths at the joint are:

$$G(x)=\vec{0}=\begin{pmatrix} g_1(x)=L_1-z_1-C_1 \\ g_2(x)=z_2-C_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.21}$$

with matrix:

$$\nabla G=\begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.22}$$

The generalized mass for this system is:

$$M=\begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{1,2}{}^{T} & M_{2,2} \end{pmatrix} \tag{4.23}$$

where each element of M is a (2×2) block matrix given by:

$$M_{1,1}=\begin{pmatrix} m_T & m_2\cos(\theta_1-\theta_2) \\ m_2\cos(\theta_1-\theta_2) & m_2 \end{pmatrix} \tag{4.24}$$

$$M_{1,2}=\begin{pmatrix} 0 & m_2 z_2\sin(\theta_1-\theta_2) \\ -m_2 L_1\sin(\theta_1-\theta_2) & 0 \end{pmatrix}=\left(M_{2,1}\right)^{T} \tag{4.25}$$

$$M_{2,2}=\begin{pmatrix} m_1(z_1)^2+m_2(L_1)^2 & m_2 L_1 z_2\cos(\theta_1-\theta_2) \\ m_2 L_1 z_2\cos(\theta_1-\theta_2) & m_2(z_2)^2 \end{pmatrix} \tag{4.26}$$

As expected the kinetic energy can be written in the form, $K = \frac{1}{2}(z,\theta)^T M(z,\theta)(z,\theta)$ and the extended generalized form of Newton's 2nd Law takes the form of a (6 × 6) system:

$$\begin{pmatrix} M & \nabla G \\ \nabla G & 0 \end{pmatrix}\begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} M_{1,1} & M_{1,2} & -1 & 0 \\ (M_{1,2})^T & M_{2,2} & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \ddot{z} \\ \ddot{\theta} \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} D \\ S\dot{\theta}^2 - \nabla P + B\dot{\theta} + Cor \\ 0 \\ 0 \end{pmatrix} \tag{4.27}$$

where,

$$D = \begin{pmatrix} (m_1 z_1 + m_2 L_1)(\dot{\theta}_1)^2 + m_2 z_2 (\dot{\theta}_2)^2 \cos(\theta_2 - \theta_1) - g m_T \sin(\theta_1) \\ m_2 L_1 (\dot{\theta}_1)^2 \cos(\theta_1 - \theta_2) + m_2 z_2 (\dot{\theta}_2)^2 - g m_2 \sin(\theta_2) \end{pmatrix} \tag{4.28}$$

The joint (1) reaction force is :

$$JRF_1 = \lambda_1 \hat{z}_1 - \lambda_2 \hat{z}_2 \tag{4.29}$$

where,

$$\begin{aligned} \lambda_1 &= -m_2 z_2 \sin(\theta_1 - \theta_2)\ddot{\theta}_2 + (m_1 z_1 + m_2 L_1)(\dot{\theta}_1)^2 + m_2 z_2 (\dot{\theta}_2)^2 \cos(\theta_2 - \theta_1) - g m_T \sin(\theta_1) \\ \lambda_2 &= -m_2 L_1 \sin(\theta_2 - \theta_1)\ddot{\theta}_1 + m_2 L_1 (\dot{\theta}_1)^2 \cos(\theta_1 - \theta_2) + m_2 z_2 (\dot{\theta}_2)^2 - g m_2 \sin(\theta_2) \end{aligned} \tag{4.30}$$

The first equation of (4.30) is identical to the reaction force of the root segment, when the generalized Equation (4.20) is applied to the case of a double pendulum. The actual contribution to the ***JRF₁*** in the $\hat{z}_1$ direction is equal and opposite to that of the reaction force at the root segment. This is consistent with the physical notion of tension in a rigid segment (Newton's 3rd Law). The second equation of (4.30) contains the terms for the pivot reaction force of the single distal pendulum (#2):

$$PRF(\text{single pendulum}) = -\lambda\hat{z} = -\left(mz\dot{\theta}^2 + gm\sin(\theta)\right)\hat{z} \tag{4.31}$$

But in addition there are two inertial force terms, namely:

$$-m_2 L_1 \sin(\theta_2 - \theta_1)\ddot{\theta}_1 + m_2 L_1 \left(\dot{\theta}_1\right)^2 \cos(\theta_1 - \theta_2) \tag{4.32}$$

Those forces arise due to the fact that the pivot of the second pendulum is not fixed to the wall but rather is moving as the result of the motion of the first pendulum. These inertial forces on the second pendulum are the components of the acceleration and centrifugal force of the first pendulum in the $\hat{z}_2$ direction.

In general, for the any two segment joint of the pendulum tree system the dynamic joint reaction force will take the form:

$$\boxed{JRF_j = \lambda_j \hat{z}_j - \lambda_{j+1}\hat{z}_{j+1}}, \tag{4.33}$$

where the subscript j refers to the proximal segment of the j^{th} joint and subscript $j+1$ refers to the distal segment of the j^{th} joint. The generalized form of the Lagrange multiplier λ_j associated with segment j of the generalized pendulum tree is presently being developed proceeding along the same lines as was used in Section 4.2-4.5 and Appendix B for $\lambda_{(1)}$ and the root segment (1).

4.8 The Relationship of the Pivot Reaction Force *(PRF)* of the Root Segment and the Ground Reaction Force *(GRF)* of a Model Walker

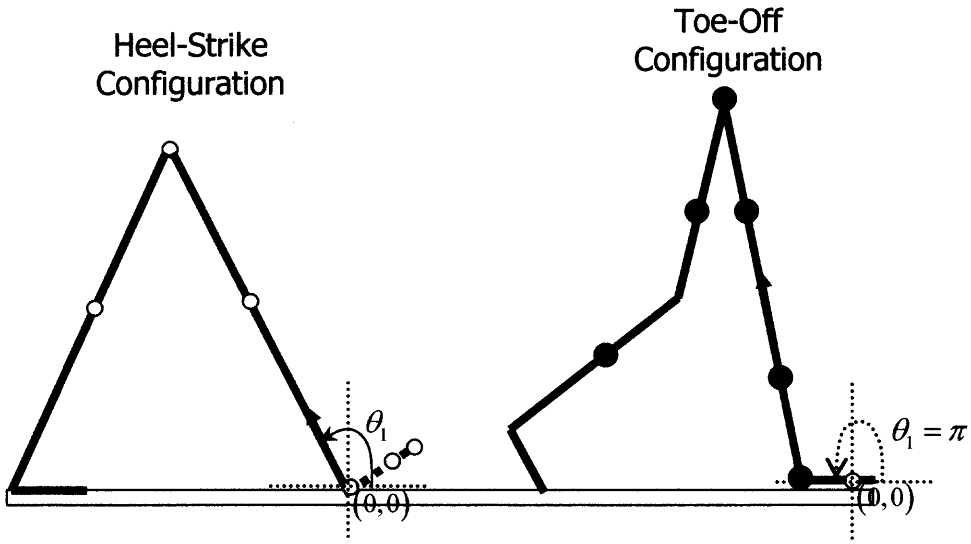

Figure 4.2 A Simple pendulum tree model of a human walker in the Heel-Strike and Toe-Off Configurations.

Figure 4.2 represents a simple idealized dynamic pendulum system model of a human walker. Assume that a force plate is under the Blue (Stance) Leg. The idealized foot of this stance leg will be assumed to have only three *potential* points of ground contact. These will be the heel, ball of the foot and toe respectively (open circles on foot of Blue Leg). During the swing phase it will be assumed that the foot-shank angle of the swing leg is not a dynamic variable and that the foot and shank can be combined and treated together as single moving unit with the foot-shank angle remaining fixed at ninety degrees. In this idealization, at the time of heel strike, there is only one point in contact with the force plate, the heel pivot point, and the ***GRF*** will essentially be the same as ***PRF*** acting along the stance leg. In Figure 4.8.1 the red arrow represents the ***PRF*** vector at the time of heel strike.

In the first model approximation the heel at this time will be considered to be the origin of the pendulum tree (the root pivot point) and the relatively small contribution of the foot of the heel strike leg on the ***PRF*** will be ignored compared to remaining segments of the body that attach to the heel of the stance leg from the shank. It will be assumed that after heel-strike the shank-foot unit will begin to rotate around the heel with the foot-shank angle remaining fixed at ninety degrees until the ball of this foot makes contact with the ground. At this time the heel-to-ball of foot will be added to the dynamical system as an additional dynamic segment (Ball of foot-Heel) and the angle between the shank and this segment will no longer be constrained to be ninety degrees but will be a dynamic variable (θ_2). At this time, the origin of the model walker will be shifted to the ball of the stance foot and the new Ball of foot-Heel segment will become the new root segment of the pendulum system.

The presence of the horizontal floor on the pendulum system will be modeled as an explicit constraint on the new root segment. Namely, it's dynamic angle, θ_1, will be constrained at the constant value of π radians keeping the Ball of foot-Heel root segment horizontal to the ground. This new constraint will generate a new generalized constraint force that will be interpreted as a reaction torque that the ground must exert at the heel to prevent the Ball of foot-Heel root segment from rotating through the floor at the root pivot point (Ball of foot)..

This generalized constraint force is assumed to be distinct from the joint reaction forces that are generated at the root pivot joint and at the heel-shank angle as these reaction forces would be present even if the floor constraint were removed. All three forces would be affecting the force plate. The joint forces acting at the Ball and Heel can be predicted using the methods described in Sections 4.6 and 4.7. The ground reaction torque maintaining the floor constraint is calculated using the concepts developed in Sections 3.5 through 3.7. As a proof of concept, this method for predicting ground reaction torque is applied in Section 4.9 to a single pendulum constrained to be horizontal on the ground and also in Section 4.10 to a double pendulum system where the root segment is forced to satisfy this same constraint.

Eventually the heel of the stance leg will lift off the ground and in this model it will be assumed that when this occurs the toe of the stance foot will simultaneously make contact with the ground. At this time, a new Toe-Ball segment will be added to the pendulum system and the origin will be moved to the Toe point making the new Toe-Ball segment the new root segment with its dynamic angle constrained at π radians to reflect its contact with the floor constraint. The previous Ball-Heel segment will become the child segment of the root (see Section 3.3) and will no longer be constrained to the ground. Its corresponding segment angle will become dynamic. The force plate will now register at the toe and ball of the foot with corresponding joint forces and torques generated at the toe and ball rather than ball and heel as before. In the last phase of the stance leg motion, the ball point of the foot will lift off the ground and the only point of contact will be the toe. During this phase the only force generated on the force plate will arise from the joint reaction force of the toe as the root pivot of the pendulum system.

In this model the center of pressure ***(COP)*** will move in discrete jumps from heel to ball of foot and finally from ball of foot to the toe of the foot as the new segments are added to the model and the origin of the pendulum system jumps from heel to ball and then from ball to toe. In summary, the walking cycle of the model will consist of 4 phases: (1) heel strike (HS) to ball strike (BS), (2) BS to heel off (HO), (3) HO to toe off (TO) and finally (TO) to (HS). In this last phase the Blue leg is the swing leg and there will be no ***GRF*** from this leg.

4.9 Proof of Concept for Ground Reaction Force (*GRF*) For Single Pendulum Resting Horizontally on the Ground

When considering a phase of the walking cycle where a segment of the foot rather than a single point is in contact with the ground then a new constraint force is introduced into the system. As described in the previous section, the presence of a foot or foot segment on the horizontal floor can be modeled as an explicit constraint on that segment of the foot in ground contact. If this segment is the root of the pendulum tree, then its dynamic angle, θ_1, will be constrained at the constant value of 0 or π radians keeping the root segment fixed horizontal to the ground. Since the floor can only provide an upward reaction force, if this constraint force changes to a downward force, this change will signal either constraint release or initiation of muscle activity to maintain the constraint.

The ideas presented in the above paragraph and in Section 4.8 can be most easily demonstrated using a single pendulum of length l resting on the ground with mass m and making contact with the ground at its pivot point (Ball of Foot or Toe) and at its end (Heel or Ball of Foot). In this case we will assume that the center of mass is also at the end of the pendulum and that the pivot point (Ball of Foot or Toe) is located at the origin and pointing in the positive x direction so that the constraint is $\theta = \pi$.

In this case θ is the superfluous coordinate. The Generalized Ground Reaction Force (*GRF*) will now be determined using Lagrange's EOM with explicit constraint. Since the single pendulum is (implicitly) constrained to move on the arc of a circle, its position s and velocity v satisfy,

$$\begin{aligned} s(t) &= l\theta(t) \\ v(t) &= l\dot{\theta}(t) \end{aligned} \tag{4.34}$$

The kinetic energy K is therefore given by,

$$K = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2 = \frac{1}{2}\dot{\theta}M\dot{\theta} \ , \tag{4.35}$$

where the generalized mass $M = ml^2$.

The generalized momentum is:

$$p = \frac{\partial K}{\partial \dot{\theta}} = M\dot{\theta} = ml^2\dot{\theta} \ , \tag{4.36}$$

and,

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial K}{\partial \dot{\theta}}\right) = \frac{\mathrm{dp}}{\mathrm{dt}} = M\ddot{\theta} = ml^2\ddot{\theta} \tag{4.37}$$

The potential energy P for the single pendulum in a constant downward gravitational field is given by:

$$P = mgy = mgl\sin\theta \ , \tag{4.38}$$

and

$$\frac{\partial L}{\partial \theta} = \frac{\partial(K-P)}{\partial \theta} - \frac{\partial P}{\partial \theta} = -\nabla P = -mgl\cos\theta \tag{4.39}$$

Lagrange's EOM:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)=\left(\frac{\partial L}{\partial \theta}\right), \tag{4.40}$$

Expressed in the generalized form of Newton's 2nd Law is:

$$M\ddot{\theta}=-\nabla P, \tag{4.41}$$

or,

$$ml^2\ddot{\theta}=-mgl\cos\theta \Rightarrow \ddot{\theta}=-\frac{g}{l}\cos\theta \tag{4.42}$$

As explained in Section 3.3, Equation (4.42) above is easily shown to be equivalent to the more familiar expression used for the frictionless single pendulum, Equation (3.01), when the relationship $\phi=\frac{\pi}{2}-\theta$ is applied between the standard polar coordinate system used in this thesis and the more conventional dynamic coordinate ϕ that measures the angle that the pendulum rod makes with the negative y-axis.

The explicit constraint function $G(\theta)=0$, that represents the pendulum (foot) resting horizontally on the ground with the pivot point (toe or ball of foot) as the origin and pointing in the positive x direction is given by:

$$G(\theta)=\theta-\pi=0 \tag{4.43}$$

Therefore,

$$\nabla G=\frac{\partial(\theta-\pi)}{\partial\theta}=1 \tag{4.44}$$

and,

$$h=\frac{\partial^2 G}{\partial\theta^2}=0 \tag{4.45}$$

Thus, the extended form of Newton's 2nd Law with explicit constraint (Equation (3.26)) is in this case given by:

$$\begin{pmatrix} M & \nabla G \\ \nabla G^T & 0 \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \lambda \end{pmatrix} = \begin{pmatrix} ml^2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \lambda \end{pmatrix} = \begin{pmatrix} -mgl\cos\theta \\ 0 \end{pmatrix} \tag{4.46}$$

Therefore:

$$\begin{gathered} ml^2\ddot{\theta} + \lambda = -mgl\cos\theta \\ \ddot{\theta} = 0 \end{gathered} \tag{4.47}$$

The generalized reaction force Equation (3.24) that the ground exerts on the stationary pendulum is,

$$F_C = -\nabla G\,\lambda\,\hat{\theta} = -\lambda\,\hat{\theta} = mgl\cos\pi\,\hat{\theta} = -mgl\,\hat{\theta} \;\; \textit{(Units of Torque)} \tag{4.48}$$

Equation (4.48) implies that the weight of the pendulum acts over a moment arm equal to the length of the rod. If the only other point of ground contact beside the pivot point is idealized as (the ball or heel of the foot) then the weight of the segment (*mg)* would be supported entirely at this point. Note that since $\theta = \pi$, the unit vector $\hat{\theta} = -\hat{y}$ points downward so that the ground reaction force F_C is pushing upward on:

$$F_C = -mgl\,\hat{\theta} = mgl\,\hat{y} \tag{4.49}$$

4.10 Proof of Concept for Ground Reaction Force (*GRF*) for Double Pendulum with Root Segment Resting on the Ground

Consider now the next simplest case, a double pendulum with the first segment resting on the ground ($\theta_1 = \pi$) but with the second segment free to rotate. As a consequence gravitational and inertial forces from the rotating segment will induce indirectly reactive forces from the ground. These will now be determined as well as the ground reaction force from the weight of the fixed root segment. The joint reaction forces for the double pendulum without any explicit constraints were determined in Section 4.7 (see Fig. 4.1 and Equations (4.29)-(4.30)). The case with explicit constraint $\theta_1 = \pi$ now being considered is illustrated below.

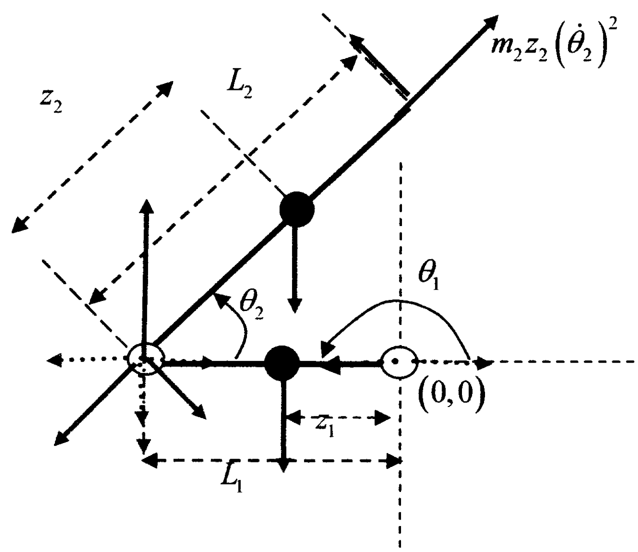

Figure 4.3 A Double pendulum system with segment 1 on the ground.

First it is significant to note that although the pivot point is (implicitly) fixed to and supported by the floor (the origin), this is not the same as the support from the floor acting to fix the dynamic angle θ_1. This is easily demonstrated by considering what would happen to the configuration illustrated in Figure 4.3 if the unconstrained double pendulum system were initially at rest and gravity were allowed to act on both mass points. The angle θ_1 would increase and the system would fall through the floor unless there is an explicit constraint added to the system to keep the first pendulum horizontal on the floor, namely, $\theta_1 = \pi$ (a constant), or

$$G(\theta_1, \theta_2) = \theta_1 - \pi = 0 \tag{4.50}$$

and,

$$\nabla G = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.51}$$

In the double pendulum system with first segment constrained by the floor the dynamic angle θ_1 plays the role of a superfluous coordinate. The generalized reaction force that maintains the floor constraint is obtained by forming the usual extended form of Newton's 2nd Law which in this case takes the form (without viscous joint forces):

$$\begin{pmatrix} M & \nabla G \\ \nabla G^T & 0 \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \lambda \end{pmatrix} = \begin{pmatrix} S\dot{\theta}^2 - \nabla P \\ 0 \end{pmatrix} \tag{4.52}$$

The first block component of Equation (4.52) reads:

$$M\ddot{\theta} = -\lambda \nabla G + S\dot{\theta}^2 - \nabla P\,, \tag{4.53}$$

which in component form is,

$$\begin{pmatrix} m_1 (z_1)^2 + m_2 (L_1)^2 & m_2 L_1 z_2 \cos(\theta_1 - \theta_2) \\ m_2 L_1 z_2 \cos(\theta_1 - \theta_2) & m_2 (z_2)^2 \end{pmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix}$$

$$= -\lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} m_2 L_1 z_2 \sin(\theta_2 - \theta_1)(\dot{\theta}_2)^2 \\ -m_2 L_1 z_2 \sin(\theta_2 - \theta_1)(\dot{\theta}_1)^2 \end{pmatrix} - g \begin{pmatrix} (m_1 z_1 + m_2 L_1)\cos(\theta_1) \\ m_2 z_2 \cos(\theta_2) \end{pmatrix} \tag{4.54}$$

Reinserting the constraint $\theta_1 = \pi \Rightarrow \dot{\theta}_1 = \ddot{\theta}_1 = 0$ gives the generalized force of constraint,

$$F_c = -\lambda \nabla G = -\lambda \hat{\theta}_1 \tag{4.55}$$

where,

$$\lambda = -m_2 L_1 z_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 + m_2 L_1 z_2 \sin(\theta_2 - \theta_1)\left(\dot{\theta}_2\right)^2 - g\left(m_1 z_1 + m_2 L_1\right)\cos\left(\theta_1\right) \tag{4.56}$$

Since $\theta_1 = \pi$, Equation (4.56) reduces to:

$$\lambda = m_2 L_1 z_2 \cos(\theta_2)\ddot{\theta}_2 - m_2 L_1 z_2 \sin(\theta_2)\left(\dot{\theta}_2\right)^2 + g\left(m_1 z_1 + m_2 L_1\right) \tag{4.57}$$

As expected from intuition the constraint force acts orthogonal to the floor. As in the case of the single pendulum the vector $\hat{\theta}_1$ points downward (-y direction) and the generalized force has units of torque. The heel contact now supports the weight of both mass points. In addition, there are inertial torque terms that arise from the angular velocity and acceleration of the second segment (shank of leg). These inertial forces are also contributing to the tension in the foot segment acting in the $\hat{z}_1$ (horizontal) direction along the floor. These are expressed in λ_1 of Equation (4.56) that represents the pivot reaction force at the root segment of a double pendulum. For $\theta_1 = \pi$, Equation (4.56) reduces to:

$$\begin{aligned} \lambda_1 &= -m_2 z_2 \sin(\theta_2)\,\ddot{\theta}_2 - \cos(\theta_2)\, m_2 z_2 \left(\dot{\theta}_2\right)^2 \\ \lambda_2 &= m_2 z_2 \left(\dot{\theta}_2\right)^2 - g m_2 \sin(\theta_2) \end{aligned} \tag{4.58}$$

The 2nd equation in (4.58) represents the magnitude of the tension on the heel joint acting along the shank in the $\hat{z}_2$ direction.

CHAPTER 5

Discussion, Future Work and Concluding Remarks

This thesis presents a theoretical framework for predicting forces of constraint that act upon the human body in motion modeled as a dynamic pendulum tree system. These constraints arise in the system due to external and internal physical factors. For example the presence of a floor is modeled as an external factor that acts upon those portions of the human pendulum system such as the heel, toe and/or ball of the stance foot that may be in contact with it. An example of internal forces that act upon the human pendulum system are the dynamic joint reaction forces that represent compression and extension forces that arise at a joint to maintain the rigid body constraints of its moving parts such as the constant length of those body segments that form the joint or that limit the anatomical range of motion of the joint. In sport additional constraints may arise on the system as a result of the 'rules of the game'. Constraints also occur necessarily as a voluntary or involuntary means by which the neuromuscular system of an individual can control and coordinate a skeletal system that without constraints represents a considerable number of dynamic degrees of freedom.

When constraints are impressed upon a dynamical system the forces they produce critically affect the dynamics of motion. For a modeling approach to find optimal techniques of movement it is necessary that it be possible to predict the constraint forces on the system and their effects on resulting dynamics, dynamic stability and energy efficiency of the motion as well as their potential to produce joint injury and/or pain.

The theoretical framework for achieving these goals is developed in this thesis and applied to predicting the ground and joint reaction forces that occur in human walking. Lagrangian dynamics and the method of superfluous coordinates are used to find constraint forces on the system when they are expressed implicitly in the equations of the model. This approach is conceptually unified with that of finding the forces of constraint when explicit constraints are added to the system. In the model developed here both approaches are required to predict ground reaction forces.

The dynamic equations developed here are for a human pendulum system moving under the influence of gravity and reacting to external constraints (the floor) and internal constraints maintaining constant segment lengths. The framework developed here can also predict the reactive forces that would be produced if additional forces were applied to the pendulum system and added to the generalized force term $F(x,\dot{x})$ in Equation (3.26). An example of this is shown in Equation (4.18) where an additional generalized non-inertial force term representing joint viscosity is included along with the generalized gravitational force in the EOM for the pendulum tree.

5.1 The Boundary Method

Extremely important non-inertial generalized force terms that arise from muscular forces acting on the human pendulum tree have not been included in the model or in this thesis. Ideally such force generating terms would simply be added to the generalized force term $F(x,\dot{x})$ in Equation (4.18) and then the effects of muscular effort on ground and joint reaction forces would be calculated by solving equations similar to Equation

(3.26) which in that case calculates the reaction force at the root pivot of the pendulum tree.

Unfortunately the programme suggested in the above paragraph can not be implemented. The appropriate net generalized muscular joint force terms are not known and in general can not even be measured experimentally at this time. EMG recordings can measure electrical muscle activity but direct non-invasive measurement of muscular force is not yet feasible. A new modeling technology that is being developed at the BME Human Performance Laboratory at NJIT, known as the Boundary Method™ represents a different approach from the programme suggested above.

This new modeling technology can solve for both human movement and the net generalized muscular joint forces that would be required to produce that movement. This modeling technology can also be applied to the theoretical framework developed in this thesis to feasibly predict the influence of muscle activity on both the joint and ground reaction forces for any human motion solution generated by the Boundary Method™. Future work will combine the Boundary Method™ technology with the theoretical framework developed in this thesis to predict joint and ground reaction forces during human walking. Although joint reaction forces can not be measured experimentally, force plates allow ground reaction forces to be measured.

The next step will compare theoretical predictions to experimental force plate measurements. The Boundary Method™ will treat each of the four phases of the walking cycle described at the end of Section 4.8 as discrete independent ballistic 2-point boundary value problems. The discontinuities that result when each phase is pieced together with its neighbors will yield the net muscular joint impulse activity from which the reactive joint and ground impulses can be calculated.

5.2 Concluding Remarks

This thesis will be used to achieve a systematic approach to predicting ground and joint reaction forces during walking. It is hoped that those predictions may in the future lead to a better understanding of the transitions between different phases of the walk, as well as understanding the walk to run transition and the triggering mechanisms that govern it. Ultimately perhaps, it will prove to provide a general methodical approach for the modeling of any human motion and the reactive forces that apply to the motion so constrained.

APPENDIX A

GENERALIZED MATRIX-VECTOR FORM OF LAGRANGE'S EOM

$M(x)\ddot{x} = F(x,\dot{x})$ (Newton's 2nd Law)

If x_i is the i^{th} generalized component of a conservative mechanical system with n degrees of freedom and no explicit constraints then Equation (3.05) represents Lagrange's equations of motion for this component:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)-\left(\frac{\partial L}{\partial x_i}\right)=0, \qquad i=1,\cdots,N \tag{A.01}$$

where $L(x,\dot{x}) = K(x,\dot{x}) - P(x)$ is the Lagrangian function for the system (Section 3.2). In this appendix it will be demonstrated that when the kinetic energy can be expressed in the form:

$$K(x,\dot{x}) = \frac{1}{2}\dot{x}^T M(x)\,\dot{x} = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\dot{x}_j M_{j,k}(x)\dot{x}_k \tag{A.02}$$

The resulting system of differential equations in (A.01) can then be expressed as a generalized form of Newton's 2nd law:

$$M(x)\ddot{x} = F(x,\dot{x}) \tag{A.03}$$

where $M(x)$ is a generalized mass (symmetric positive definite) matrix, $F(x,\dot{x})$ a generalized force vector and $\ddot{x}$ is a generalized acceleration vector.

Since the potential energy P depends explicitly on the configuration x of the system but not on the generalized velocity $\dot{x}$, Equation (A.03) can be written as:

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}_i}\right)=\left(\frac{\partial K}{\partial x_i}\right)-\frac{\partial P}{\partial x_i}, \qquad i=1,\cdots,N \tag{A.04}$$

Differentiating Equation (A.02) with respect to $\dot{x}_i$ and applying the product rule gives the following results:

$$\frac{\partial K}{\partial \dot{x}_i} = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial \dot{x}_j}{\partial \dot{x}_i} M_{j,k}(x)\dot{x}_k + \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\dot{x}_j M_{j,k}(x)\frac{\partial \dot{x}_k}{\partial \dot{x}_i} \qquad \text{(A.05)}$$

Since $\frac{\partial \dot{x}_j}{\partial \dot{x}_i} = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$, the double sums of Equation (A.05) can be replaced by single sums:

$$\frac{\partial K}{\partial \dot{x}_i} = \frac{1}{2}\sum_{k=1}^{N} M_{i,k}(x)\dot{x}_k + \frac{1}{2}\sum_{j=1}^{N}\dot{x}_j M_{j,i}(x) \qquad \text{(A.06)}$$

Replacing the dummy variable j in the second term with k and using the symmetry of M shows that both sums in Equation (A.06) are the same and therefore:

$$\frac{\partial K}{\partial \dot{x}_i} = \sum_{k=1}^{N} M_{i,k}(x)\dot{x}_k = (M\dot{x})_i = p_i \qquad \text{(A.07)}$$

The vector p is defined as the generalized momentum of the system. The left hand side of Equation (A.01) is therefore:

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}_i}\right) = \frac{dp_i}{dt} = \frac{d}{dt}(M\dot{x})_i = (M\,\ddot{x})_i + (\frac{dM}{dt}\dot{x})_i \qquad \text{(A.08)}$$

Therefore the equivalent form of Lagrange's Equations of Motion can be expressed as:

$$(M\,\ddot{x})_i = \left(\frac{\partial K}{\partial x_i}\right) - (\frac{dM}{dt}\dot{x})_i - \frac{\partial P}{\partial x_i}, \qquad i = 1, \cdots, N \qquad \text{(A.09)}$$

The above equation in matrix form yields the desired result:

$$
\begin{aligned}
&M(x)\,\ddot{x} = F(x,\dot{x}) = s(x,\dot{x}) - \nabla P(x) \\
&\left(\nabla P(x)\right)_i = \frac{\partial P}{\partial x_i} \\
&s_i(x,\dot{x}) = \left(\frac{\partial K(x,\dot{x})}{\partial x_i}\right) - \left(\frac{dM(x)}{dt}\dot{x}\right)_i
\end{aligned}
\tag{A.10}
$$

The vector $s(x,\dot{x})$ in Equation (A.10) is a generalized inertial force that can be further analyzed since,

$$
\frac{\partial K}{\partial x_i} = \frac{1}{2}\dot{x}^T \frac{\partial M(x)}{\partial x_i}\dot{x} = \frac{1}{2}\sum_{j=1}^{N}\dot{x}_j \sum_{k=1}^{N}\frac{\partial M_{j,k}(x)}{\partial x_i}\dot{x}_k \tag{A.11}
$$

and

$$
\left(\frac{dM}{dt}\dot{x}\right)_i = \sum_{j=1}^{N}\frac{dM_{i,j}}{dt}\dot{x}_j \tag{A.12}
$$

Applying the chain rule to Equation (A.12) yields,

$$
\left(\frac{dM}{dt}\dot{x}\right)_i = \sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial M_{i,j}(x)}{\partial x_k}\dot{x}_k\dot{x}_j \tag{A.13}
$$

Substituting Equation (A.11) and (A.13) into s_i of Equation (A.10) gives,

$$
s_i(x,\dot{x}) = \frac{\partial K}{\partial x_i} - \left(\frac{dM}{dt}\dot{x}\right)_i = -\sum_{j=1}^{N}\sum_{k=1}^{N}\left(\frac{\partial M_{i,j}(x)}{\partial x_k} - \frac{1}{2}\frac{\partial M_{j,k}(x)}{\partial x_i}\right)\dot{x}_j\dot{x}_k \tag{A.14}
$$

As an example, Equation (A.14) will be applied to the generalized 2-D branched pendulum system of Section 3.3 where:

$$
K = \frac{1}{2}\sum_{i=1}^{P} m_i v_i^2 = \frac{1}{2}\dot{\theta}^T M(\theta)\,\dot{\theta}\,, \tag{A.15}
$$

and the components of the generalized mass matrix are ,

$$
M_{i,j} = C_{i,j}\cos(\theta_i - \theta_j)\,, \tag{A.16}
$$

where C is the symmetric matrix,

$$C = R^T M_D R, \tag{A.17}$$

$$\mathbf{M}_D = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & m_P \end{bmatrix}, \tag{A.18}$$

and R is the relation matrix defined in Section 3.3. In this case,

$$\frac{\partial M_{i,j}(x)}{\partial x_k} = \frac{\partial C_{i,j}\cos(\theta_i - \theta_j)}{\partial \theta_k} = \begin{cases} -C_{i,j}\sin(\theta_i - \theta_j), & k = i \\ C_{i,j}\sin(\theta_i - \theta_j), & k = j \\ 0, & k \neq i \; or \; j \end{cases}, \tag{A.19}$$

thus,

$$-\sum_{j=1}^{N}\sum_{k=1}^{N}\left(\frac{\partial M_{i,j}(x)}{\partial x_k}\right)\dot{x}_j\dot{x}_k = \sum_{j=1}^{N} C_{i,j}\sin(\theta_j - \theta_i)\dot{\theta}_j(\dot{\theta}_j - \dot{\theta}_i) \tag{A.20}$$

In a similar fashion to the method used to obtain Equation (A.19) it can be shown that:

$$\frac{\partial M_{j,k}(x)}{\partial x_i} = \frac{\partial C_{j,k}\cos(\theta_j - \theta_k)}{\partial \theta_i} = \begin{cases} -C_{i,k}\sin(\theta_i - \theta_k), & i = j \\ C_{j,i}\sin(\theta_j - \theta_i), & i = k \\ 0, & i \neq j \; or \; k \end{cases} \tag{A.21}$$

Since,

$$\begin{aligned} &\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial M_{j,k}(x)}{\partial x_i}\dot{x}_j\dot{x}_k = \\ &\frac{1}{2}\sum_{j=1}^{N}\frac{\partial M_{j,i}(x)}{\partial x_i}\dot{x}_j\dot{x}_i\bigg|_{k=i} + \frac{1}{2}\sum_{k=1}^{N}\frac{\partial M_{i,k}(x)}{\partial x_i}\dot{x}_i\dot{x}_k\bigg|_{j=i} + \frac{1}{2}\sum_{j\neq i}\sum_{k\neq i}\frac{\partial M_{j,k}(x)}{\partial x_i}\dot{x}_j\dot{x}_k \end{aligned} \tag{A.22}$$

From Equation (A.21) the last term in Equation (A.22) is 0. Substituting Equation (A.21) into each of two remaining terms of Equation (A.22) gives:

$$\frac{1}{2}\sum_{k=1}^{N}\frac{\partial M_{i,k}(x)}{\partial x_i}\dot{x}_i\dot{x}_k\bigg|_{j=i} = -\frac{1}{2}\sum_{k=1}^{N}C_{i,k}\sin(\theta_i-\theta_k)\dot{\theta}_i\dot{\theta}_k = \frac{1}{2}\sum_{j=1}^{N}C_{i,j}\sin(\theta_j-\theta_i)\dot{\theta}_i\dot{\theta}_j \qquad \text{(A.23)}$$

and,

$$\frac{1}{2}\sum_{j=1}^{N}\frac{\partial M_{j,i}(x)}{\partial x_i}\dot{x}_j\dot{x}_i\bigg|_{k=i} = \frac{1}{2}\sum_{j=1}^{N}C_{j,i}\sin(\theta_j-\theta_i)\dot{\theta}_i\dot{\theta}_j = \frac{1}{2}\sum_{j=1}^{N}C_{i,j}\sin(\theta_j-\theta_i)\dot{\theta}_i\dot{\theta}_j \qquad \text{(A.24)}$$

Thus:

$$\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial M_{j,k}(x)}{\partial x_i}\dot{x}_j\dot{x}_k = \sum_{j=1}^{N}C_{i,j}\sin(\theta_j-\theta_i)\dot{\theta}_i\dot{\theta}_j \qquad \text{(A.25)}$$

Substituting Equations (A.25) and (A.20) into Equation (A.14) gives:

$$s_i(x,\dot{x}) = -\sum_{j=1}^{N}\sum_{k=1}^{N}\left(\frac{\partial M_{i,j}(x)}{\partial x_k} - \frac{1}{2}\frac{\partial M_{j,k}(x)}{\partial x_i}\right)\dot{x}_j\dot{x}_k = \sum_{j=1}^{N}C_{i,j}\sin(\theta_j-\theta_i)\left(\dot{\theta}_j\right)^2 \qquad \text{(A.26)}$$

which in matrix-vector form is:

$$s = S\dot{\theta}^2 \qquad \text{(A.27)}$$

where S is the skew–symmetric matrix with components,

$$S_{i,j} = C_{i,j}\sin(\theta_j-\theta_i) \quad i,j = 1,\cdots,N \qquad \text{(A.28)}$$

Substituting Equation (A.27) into Equation (A.10) gives Equation (3.17) that was used for the generalized 2-D branched pendulum tree system in Section 3.3.

APPENDIX B

DERIVATIONS

Part 1: Pivot Reaction Force

The derivation will proceed by breaking down the equation into its sub-components:

$$\left(\frac{\partial L}{\partial \dot{z}_1}\right)=\frac{\partial K}{\partial \dot{z}_1}-\frac{\partial P}{\partial \dot{z}_1} \tag{B.1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_1}\right)=\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{z}_1}\right)-\frac{d}{dt}\left(\frac{\partial P}{\partial \dot{z}_1}\right) \tag{B.2}$$

$$\left(\frac{\partial L}{\partial z_1}\right)=\frac{\partial K}{\partial z_1}-\frac{\partial P}{\partial z_1} \tag{B.3}$$

Equation (B.1) can be written as:

$$\left(\frac{\partial L}{\partial \dot{z}_1}\right)=\frac{\partial K}{\partial \dot{z}_1}-\frac{\partial P}{\partial \dot{z}_1}=\left(\frac{\partial K_o}{\partial \dot{z}_1}+\frac{\partial K_n}{\partial \dot{z}_1}\right)-\frac{\partial P}{\partial \dot{z}_1}, \tag{B.4}$$

where K_o denotes the kinetic energy of the system without the superfluous coordinate and is given by Equation (3.14) and $K_n=\frac{1}{2}m_T\,\dot{z}_1^2+\dot{z}_1 u^T\dot{\theta}$ are additional terms due to the superfluous coordinate. The vector u and scalar m_T are defined in Equation (4.11) and Equation (4.10) respectively. Note that in Equation (B.1) the potential energy does not depend upon $\dot{z}_1$, therefore, $\frac{\partial P}{\partial \dot{z}_1}=0$. Furthermore the *original* kinetic energy terms are also independent of $\dot{z}_1$, thus, Equation (B.1) can be expressed as:

$$\left(\frac{\partial L}{\partial \dot{z}_1}\right)=\left(\frac{\partial K_n}{\partial \dot{z}_1}\right)=\frac{\partial(\frac{1}{2}m_T\dot{z}_1^2+\dot{z}_1u^T\dot{\theta})}{\partial \dot{z}_1}=m_T\dot{z}_1+u^T\dot{\theta} \tag{B.5}$$

where,,

$$u_j=\left(\underline{M}^TR\right)_j\sin(\theta_1-\theta_j),\quad j=1,\ldots,S \tag{B.6}$$

and

$$\left(\underline{M}^TR\right)_j=\sum_{i=1}^{p}m_iR_{ij},\quad j=1,\cdots,S \tag{B.7}$$

Equation (B.2) can be obtained by taking the derivative of Equation (B.5) with respect to time, resulting in:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_1}\right)=m_T\ddot{z}_1+\left(\frac{d}{dt}u^T\right)\dot{\theta}+u^T\ddot{\theta} \tag{B.8}$$

$$\left(\frac{d}{dt}u^T\right)_i=\dot{u}_i=\left(\underline{M}^TR\right)_i\cos\left(\theta_1-\theta_i\right)\left(\dot{\theta}_1-\dot{\theta}_i\right)+\sin\left(\theta_1-\theta_i\right)\frac{d}{dt}\left(\underline{M}^TR\right)_i \tag{B.9}$$

Since,

$$\frac{d}{dt}\left(\underline{M}^TR\right)_i=\frac{\partial}{\partial z_1}\left(\underline{M}^TR_i\right)\dot{z}_1, \tag{B.10}$$

and,

$$\frac{\partial}{\partial z_1}\left(\underline{M}^TR_i\right)=\begin{cases}M_T, & i=1\\ 0, & otherwise\end{cases} \tag{B.11}$$

Then the second term on the RHS of Equation (B.9), $\sin\left(\theta_1-\theta_i\right)\frac{d}{dt}\left(\underline{M}^TR\right)_i=0$, and

$$\dot{u}_i=\left(\underline{M}^TR\right)_i\cos\left(\theta_1-\theta_i\right)\left(\dot{\theta}_1-\dot{\theta}_i\right) \tag{B.12}$$

$$\left(\frac{d}{dt}u^T\right)\dot{\theta}=\sum_{i=1}^{S}\left(\underline{M}^TR\right)_i\cos\left(\theta_1-\theta_i\right)\dot{\theta}_1\dot{\theta}_i-\sum_{i=1}^{S}\left(\underline{M}^TR\right)_i\cos\left(\theta_1-\theta_i\right)\dot{\theta}_i^2 \tag{B.13}$$

By defining the vector w with components given by:

$$w_i = \left(\underline{M}^T R\right)_i \cos\left(\theta_1 - \theta_i\right), \quad i = 1, \cdots, S \tag{B.14}$$

Equation (B.13) can be written in the form:

$$\left(\frac{d}{dt}u^T\right)\dot{\theta} = \dot{\theta}_1 w^T \dot{\theta} - w^T \dot{\theta}^2 \tag{B.15}$$

Substituting Equation (B.15) into Equation (B.8) yields:

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_1}\right) = m_T \ddot{z}_1 + u^T \ddot{\theta} + \dot{\theta}_1 w^T \dot{\theta} - w^T \dot{\theta}^2} \tag{B.16}$$

where w is a vector whose components are given by:

$$w_i = \left(\underline{M}^T R\right)_i \cos\left(\theta_1 - \theta_i\right), \quad i = 1, \cdots, S \tag{B.17}$$

Substituting Equation (4.09) into Equation (B.3) will yield the result:

$$\frac{\partial K}{\partial z_1} = \frac{\partial K_o}{\partial z_1} + \dot{z}_1 \frac{\partial\left(u^T\right)}{\partial z_1}\dot{\theta} \tag{B.18}$$

Using Equation (B.11), it is easily shown that $\dfrac{\partial\left(u^T\right)}{\partial z_1} = 0$ and therefore:

$$\frac{\partial K}{\partial z_1} = \frac{\partial K_o}{\partial z_1} \tag{B.19}$$

Substituting Equation (3.14) into Equation (B.19) above gives:

$$\frac{\partial K}{\partial z_1} = \frac{1}{2}\dot{\theta}^T \frac{\partial M}{\partial z_1}\dot{\theta} \tag{B.20}$$

where the components of $\dfrac{\partial M}{\partial z_1}$ are given by:

$$\frac{\partial M_{i,j}}{\partial z_1} = \frac{\partial C_{i,j}\cos(\theta_i - \theta_j)}{\partial z_1} = \cos(\theta_i - \theta_j)\frac{\partial C_{i,j}}{\partial z_1}, \tag{B.21}$$

and,

$$\frac{\partial C_{i,j}}{\partial z_1} = \frac{\partial}{\partial z_1}\left(R^T M_D R\right) = \sum_{k=1}^{P} m_k \frac{\partial\left(R_{ki} R_{kj}\right)}{\partial z_1} = \sum_{k=1}^{P} m_k R_{ki} \frac{\partial R_{kj}}{\partial z_1} + \sum_{k=1}^{P} m_k R_{kj} \frac{\partial R_{ki}}{\partial z_1} \tag{B.22}$$

Substituting Equations (B.21) and (B.22) into Equation (B.20) gives:

$$\frac{\partial K}{\partial z_1} = \Psi_1 + \Psi_2, \tag{B.23}$$

where,

$$\Psi_1 = \frac{1}{2}\sum_{i=1}^{S} \dot{\theta}_i \sum_{j=1}^{S} \dot{\theta}_j \cos\left(\theta_i - \theta_j\right) \sum_{k=1}^{P} m_k R_{ki} \frac{\partial R_{kj}}{\partial z_1}, \tag{B.24}$$

and

$$\Psi_2 = \frac{1}{2}\sum_{i=1}^{S} \dot{\theta}_i \sum_{j=1}^{S} \dot{\theta}_j \cos\left(\theta_i - \theta_j\right) \sum_{k=1}^{P} m_k R_{kj} \frac{\partial R_{ki}}{\partial z_1} \tag{B.25}$$

Since the root segment of the tree is a forefather to all other branches, every length in the first column of R will change with the superfluous coordinate z_1 but all lengths in the other columns of R will be independent of z_1 (Section 3.3 and Example in Figure 3.2 with explicit R given by Equation (3.13)). More precisely:

$$\frac{\partial R_{kj}}{\partial z_1} = \begin{cases} 1, & j = 1; \quad k = 1, \cdots, P \\ 0, & j \neq 1; \quad k = 1, \cdots, P \end{cases} \tag{B.26}$$

umming Equation (B.24) over j yields:

$$\Psi_1 = \frac{1}{2}\dot{\theta}_1 \sum_{i=1}^{S} \dot{\theta}_i \cos\left(\theta_i - \theta_1\right) \sum_{k=1}^{P} m_k R_{ki}, \tag{B.27}$$

or

$$\Psi_1 = \frac{1}{2}\dot{\theta}_1 w^T \dot{\theta} \tag{B.28}$$

A similar argument can be used to show that $\Psi_2 = \Psi_1 = \frac{1}{2}\dot{\theta}_1 w^T \dot{\theta}$, therefore:

$$\frac{\partial K}{\partial z_1} = \Psi_1 + \Psi_2 = \dot{\theta}_1 w^T \dot{\theta} \tag{B.29}$$

The potential energy expressed by Equation (3.16) without the superfluous coordinate is the same as with it, and therefore:

$$\frac{\partial P}{\partial z_1} = \frac{\partial P_o}{\partial z_1} = m_T g \sin\theta_1 \tag{B.30}$$

Substitution of Equations(B.30) and (B.29) into (B.3) yields:

$$\boxed{\left(\frac{\partial L}{\partial z_1}\right) = \frac{\partial K}{\partial z_1} - \frac{\partial P}{\partial z_1} = \dot{\theta}_1 w^T \dot{\theta} - m_T g \sin\theta_1} \tag{B.31}$$

Combing Equations(B.31) and (B.16) yields the z_1-component of Lagrange's EOM in the $\hat{z}_1$ direction, namely:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_1}\right) = m_T \ddot{z}_1 + u^T \ddot{\theta} + \dot{\theta}_1 w^T \dot{\theta} - w^T \dot{\theta}^2 = \left(\frac{\partial L}{\partial z_1}\right) = \frac{\partial K}{\partial z_1} - \frac{\partial P}{\partial z_1} = \dot{\theta}_1 w^T \dot{\theta} - m_T g \sin\theta_1,$$

or equivalently:

$$m_T \ddot{z}_1 + u^T \ddot{\theta} = w^T \dot{\theta}^2 - m_T g \sin\theta_1 \tag{B.32}$$

This is the first block of Equation (4.15).

When the constraint that z_1 is constant is inserted into the extended form of Equation (4.18) then the first block becomes Equation (4.19) which for convenience is reproduced below:

$$\begin{pmatrix} m_T & u^T & 1 \end{pmatrix} \begin{pmatrix} \ddot{z}_1 \\ \ddot{\theta} \\ \lambda \end{pmatrix} = w^T \dot{\theta}^2 - g m_T \sin(\theta_1) \tag{B.33}$$

Solving Equation (B.33) for λ reproduces Equation (4.20) for the *PRF* at the root of the pendulum tree:

$$\boxed{\begin{aligned} & -\lambda \hat{z}_1 = \left(u^T \ddot{\theta} - w^T \dot{\theta}^2 + m_T g \sin\theta_1\right) \hat{z}_1 \\ & \textit{where,} \\ & m_T = \sum_{i=1}^{P} m_i \\ & \underline{M}^T = (m_1, m_2, \ldots, m_P) \\ & u_i = \left(\underline{M}^T R\right)_i \sin(\theta_1 - \theta_i), \quad i = 1, \cdots, S \\ & w_i = \left(\underline{M}^T R\right)_i \cos(\theta_1 - \theta_i), \quad i = 1, \cdots, S \end{aligned}} \tag{B.34}$$

Part 2: Pendulum Tree in Flight

To derive the equations for the pendulum tree in flight, it is necessary to obtain the new terms for the θ_i-component of Lagrange's EOM (Equation (3.14)) in each of the $\hat{\theta}_i$-directions that result when z_1 is considered to be dynamic, that is, as function of *t*.

The kinetic energy function with dynamic z_1 is given by:

$$K = K_0 + K_n, \tag{B.35}$$

where,

$$K_n = \frac{1}{2} m_T \dot{z}_1^2 + \dot{z}_1 u^T \dot{\theta} \tag{B.36}$$

and where K_0 is the kinetic energy of the pendulum tree with z_1 implicitly constrained. There are no new terms in the potential energy except that z_1 is now dynamic, so $P = P_0$ and:

$$(\nabla P)_i \equiv \frac{\partial P}{\partial \theta_i} = (\nabla P_0)_i = \frac{\partial P_0}{\partial \theta_i} = g\left(\underline{M}^T R\right)_i \cos(\theta_i), \tag{B.37}$$

which is Equation (3.16).

The θ_i-component of Lagrange's EOM (Equation (3.14)) is:

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}_i}\right) = \frac{\partial K}{\partial \theta_i} - \frac{\partial P}{\partial \theta_i}, \qquad i = 1, \cdots, S \tag{B.38}$$

Equation (B.38) for the θ_i-component of a pendulum tree with z_1 implicitly constrained can be expressed in the form of Equation (3.17) which for convenience is duplicated here as:

$$\left(M\ddot{\theta}\right)_i = \left(S(\theta)\,\dot{\theta}^2\right)_i - (\nabla P)_i \tag{B.39}$$

Applying Equations(B.35) and (B.37) to Equation (B.38) shows that the EOM for the θ_i-component of pendulum tree with z_1 dynamic contains EOM for the θ_i-component of pendulum tree with z_1 constrained but there are additional terms to be added to Equation (B.39) that arise in part from the added kinetic energy terms in K_n (Equation (B.36). More precisely, the new terms in Equation (B.39) that arise from K_n are:

$$\left(M\ddot{\theta}\right)_i + \frac{d}{dt}\left(\frac{\partial K_n}{\partial \dot{\theta}_i}\right) = \left(S(\theta)\,\dot{\theta}^2\right)_i - \left(\nabla P\right)_i + \frac{\partial K_n}{\partial \theta_i} \tag{B.40}$$

But there is an additional term that must be added to Equation (B.40) that arises from $\frac{d}{dt}\left(\frac{\partial K_0}{\partial \dot{\theta}_i}\right)$ and is not considered in Equation (B.39). Although the expression for the generalized momentum $\frac{\partial K_0}{\partial \dot{\theta}_i}$ is the same with or without the implicit constraint on z_1, an additional term is present when the derivative of the generalized momentum is taken with respect to time due to dynamic nature of z_1. More precisely the additional term A_{ni} is given by:

$$A_{ni} = \frac{1}{2}\frac{\dot{\theta}^T \frac{d}{dt}\left(M\left(z_1,\theta\right)\right)\dot{\theta}}{\partial \dot{\theta}_i} \tag{B.41}$$

Therefore three terms, $\frac{\partial K_n}{\partial \theta_i}, \frac{d}{dt}\left(\frac{\partial K_n}{\partial \dot{\theta}_i}\right)$ and A_{ni} must be determined and added to Equation (B.39) to obtain the θ_i-component of pendulum tree with z_1 dynamic,

$$\left(M\ddot{\theta}\right)_i + \frac{d}{dt}\left(\frac{\partial K_n}{\partial \dot{\theta}_i}\right) + A_{ni} = \left(S(\theta)\,\dot{\theta}^2\right)_i - \left(\nabla P\right)_i + \frac{\partial K_n}{\partial \theta_i} \tag{B.42}$$

Expressions for each of the three terms, $\frac{\partial K_n}{\partial \theta_i}, \frac{d}{dt}\left(\frac{\partial K_n}{\partial \dot{\theta}_i}\right)$ and A_{ni} will now be considered. In what follows extensive use will be made of the Kronecker- δ -Function,

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & otherwise \end{cases} \tag{B.43}$$

and its property,

$$\sum_{j=1}^{N} f(x_j)\,\delta_{ij} = f(x_i) \tag{B.44}$$

Consider first the added term to Equation(B.42), $\frac{\partial K_n}{\partial \theta_i}$. Since the vector u in Equation (B.34)depends upon θ_i ,

$$\frac{\partial K_n}{\partial \theta_i} = \dot{z}_1 \frac{\partial u^T}{\partial \theta_i} \dot{\theta}, \tag{B.45}$$

where the dot product is:

$$\left(\frac{\partial u^T}{\partial \theta_i}\right)\dot{\theta} = \sum_{j=1}^{S} \frac{\partial u_j}{\partial \theta_i} \dot{\theta}_j \tag{B.46}$$

$$\left(\frac{\partial u^T}{\partial \theta_i}\right)\dot{\theta} = \sum_{j=1}^{S} \frac{\partial \left(\underline{M}^T R\right)_j \sin\left(\theta_1 - \theta_j\right)}{\partial \theta_i} \dot{\theta}_j = -\sum_{j=1}^{S} w_j \frac{\partial \theta_j}{\partial \theta_i} \dot{\theta}_j \tag{B.47}$$

and w is the vector defined in Equation (B.34). Since,

$$\frac{\partial \theta_j}{\partial \theta_i} = \delta_{i,j}, \tag{B.48}$$

then employing the property expressed in Equation (B.44) yields:

$$\frac{\partial K_n}{\partial \theta_i} = -\dot{z}_1 w_i \dot{\theta}_i \tag{B.49}$$

Now consider the second added term to Equation (B.42), $\frac{d}{dt}\left(\frac{\partial K_n}{\partial \dot{\theta}_i}\right)$. Differentiating Equation (B.36) yields:

$$\frac{\partial K_n}{\partial \dot{\theta}_i} = \dot{z}_1 u^T \frac{\partial \dot{\theta}}{\partial \dot{\theta}_i}, \tag{B.50}$$

where,

$$\left(\frac{\partial \dot{\theta}}{\partial \dot{\theta}_i}\right)_j = \frac{\partial \dot{\theta}_j}{\partial \dot{\theta}_i} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & otherwise \end{cases} \tag{B.51}$$

Applying the property expressed in Equation (B.44), to Equation (B.50) $u^T \frac{\partial \dot{\theta}}{\partial \dot{\theta}_i} = u_i$ and,

$$\frac{\partial K_n}{\partial \dot{\theta}_i} = \dot{z}_1 u_i \tag{B.52}$$

Differentiating Equation (B.52)with respect to time gives:

$$\frac{d}{dt}\left(\frac{\partial K_n}{\partial \dot{\theta}_i}\right) = \ddot{z}_1 u_i + \dot{z}_1 \dot{u}_i \tag{B.53}$$

Using Equations (B.12) and (B.14) gives $\dot{u}_i = w_i\left(\dot{\theta}_1 - \dot{\theta}_i\right)$ and this results in the new term, $\frac{d}{dt}\left(\frac{\partial K_n}{\partial \dot{\theta}_i}\right)$ to be evaluated as:

$$\frac{d}{dt}\left(\frac{\partial K_n}{\partial \dot{\theta}_i}\right) = \ddot{z}_1 u_i + \dot{z}_1 w_i \left(\dot{\theta}_1 - \dot{\theta}_i\right) \tag{B.54}$$

Finally consider the third added term to Equation (B.42), A_{ni}. Writing Equation (B.41) in component form and using Equation (3.14) for $M\left(z_1, \theta\right)$ gives:

$$A_{ni} = \frac{1}{2}\sum_{j=1}^{S}\sum_{k=1}^{S}\frac{dC_{j,k}(z_1)}{dt}\cos(\theta_j - \theta_k)\frac{\partial}{\partial \dot{\theta}_i}\left(\dot{\theta}_j \dot{\theta}_k\right), \tag{B.55}$$

where,

$$\frac{dC_{j,k}(z_1)}{dt} = \frac{d}{dt}\sum_{l=1}^{P} R_{lj}(z_1)m_l R_{lk}(z_1) = \dot{z}_1\sum_{l=1}^{P}\frac{d}{dz_1}\left[R_{lj}(z_1)m_l R_{lk}(z_1)\right]$$
$$= \dot{z}_1\sum_{l=1}^{P}\left[\frac{dR_{lj}(z_1)}{dz_1}m_l R_{lk}(z_1) + R_{lj}(z_1)m_l\frac{dR_{lk}(z_1)}{dz_1}\right] \tag{B.56}$$

Therefore, the new terms can be decomposed into two similar terms,

$$A_{ni} = A_{1ni} + A_{2ni}, \tag{B.57}$$

where,

$$A_{1ni} = \frac{1}{2}\dot{z}_1\sum_{j=1}^{S}\sum_{k=1}^{S}\sum_{l=1}^{P}\frac{dR_{lj}(z_1)}{dz_1}m_l R_{lk}(z_1)\cos(\theta_j-\theta_k)\left(\frac{\partial}{\partial\dot{\theta}_i}\left(\dot{\theta}_j\dot{\theta}_k\right)\right), \tag{B.58}$$

and,

$$A_{2ni} = \frac{1}{2}\dot{z}_1\sum_{j=1}^{S}\sum_{k=1}^{S}\sum_{l=1}^{P}\frac{dR_{lk}(z_1)}{dz_1}m_l R_{lj}(z_1)\cos(\theta_j-\theta_k)\left(\frac{\partial}{\partial\dot{\theta}_i}\left(\dot{\theta}_j\dot{\theta}_k\right)\right) \tag{B.59}$$

Using the fact that,

$$\frac{dR_{lj}(z_1)}{dz_1} = \begin{cases}1, & j=1; l=1,\cdots,P\\ 0, & j\neq 1; l=1,\cdots,P\end{cases}$$
$$\frac{dR_{lk}(z_1)}{dz_1} = \begin{cases}1, & k=1; l=1,\cdots,P\\ 0, & k\neq 1; l=1,\cdots,P\end{cases} \tag{B.60}$$

and summing A_{1ni} first over j and summing A_{2ni} first over k shows that except for a dummy variable $A_{1ni} = A_{2ni}$. Therefore $A_{ni} = A_{1ni} + A_{2ni} = 2A_{1ni}$ or

$$A_{ni} = \dot{z}_1\sum_{j=1}^{S}\sum_{k=1}^{S}\sum_{l=1}^{P}\frac{dR_{lj}(z_1)}{dz_1}m_l R_{lk}(z_1)\cos(\theta_j-\theta_k)\left(\frac{\partial}{\partial\dot{\theta}_i}\left(\dot{\theta}_j\dot{\theta}_k\right)\right) \tag{B.61}$$

Summing Equation (B.62) above first over l and then over j gives:

$$A_{ni} = \dot{z}_1\sum_{k=1}^{S}\left(\underline{M}^T R\right)_k\cos(\theta_1-\theta_k)\dot{\theta}_1\left(\frac{\partial}{\partial\dot{\theta}_i}\left(\dot{\theta}_k\right)\right) = \dot{\theta}_1 w_i \tag{B.63}$$

Finally, summing over k and using,

$$\frac{\partial}{\partial \dot{\theta}_i}\left(\dot{\theta}_k\right) = \delta_{i,k}, \tag{B.64}$$

produces the result:

$$A_{ni} = \dot{z}_1 \dot{\theta}_1 w_i \tag{B.65}$$

Substituting Equations (B.49), (B.54) and (B.65) for the three terms, $\frac{\partial K_n}{\partial \theta_i}$, $\frac{d}{dt}\left(\frac{\partial K_n}{\partial \dot{\theta}_i}\right)$ in Equation (B.42) yields the θ_i-component of pendulum tree with z_1 dynamic:

$$\left(M\ddot{\theta}\right)_i + \ddot{z}_1 u_i = \left(S(\theta)\,\dot{\theta}^2\right)_i - \left(\nabla P\right)_i - 2\dot{z}_1\dot{\theta}_1 w_i, \quad i = 1,\cdots,S, \tag{B.66}$$

where the term $-2\dot{z}_1\dot{\theta}_1 w_i = Cor$ represents the generalized Coriolis force.

Combining Equation (B.66) with Equation (B.33) for the $\hat{z}_1$ direction produces the full set of equations that describe the 2-D pendulum tree in flight. This is given by:

$$\boxed{\begin{pmatrix} m_T & u^T \\ u & M \end{pmatrix}\begin{pmatrix} \ddot{z}_1 \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} w^T\dot{\theta}^2 - gm_T \sin\left(\theta_1\right) \\ S\dot{\theta}^2 - \nabla P + B\dot{\theta} + Cor \end{pmatrix}}, \tag{B.67}$$

where the matrix B has components that represent joint viscosity coefficients (Section 3.3).

When the constraint $z_1 = const$ is explicitly reinforced then the extended form Equation (4.18) is recovered from Equation (B.67).

REFERENCES

Alexander, R. M. (1989). "Optimization and gaits in the locomotion of vertebrates." *Physiol. Rev.* 69, 1199-1227.

Beaupied, H., Multon, F. and Delamarche, P. (2003). "Does training have consequences for the walk–run transition speed?" *Hum. Mov. Sci.* 22, 1-12.

Getchell, N. and Whitall, J. (1997). "Transitions in gait as a function of physical parameters (abstract)." *J. Sport Exercise Psychol.* 19, S55.

Richard R. Neptune and Kotaro Sasaki (2004). "Ankle plantar flexor force production is an important determinant of the preferred walk-to-run transition speed." *The University of Texas at Austin, Austin, TX 78712, USA* 2004.

Hanna, A., Abernethy, B., Neal, R. J. and Burgess-Limerick, R. (2000). "Triggers for the transition between human walking and running." *In Energetics of Human Activity* (ed. W. A. Sparro), pp. 124-164.

Chicago: Human Kinetic Kram, R. (2000). "Muscular force or work: what determines the metabolic energy cost of running?" *Exerc. Sport Sci. Rev.* 28, 138-143.

Li, L. and Hamill, J. (2002). "Characteristics of the vertical ground reaction force component prior to gait transition." *Res. Q. Exerc. Sport* 73, 229-237.

Raynor, A. J., Yi, C. J., Abernethy, B. and Jong, Q. J. (2002). "Are transitions in human gait determined by mechanical, kinetic or energetic factors?"*Hum. Mov. Sci.* 21, 785-805.

Farley, C. T. and Ferris, D. P. (1998). "Biomechanics of walking and running: from center of mass movement to muscle action." *Exerc.Sport Sci. Rev.* 26, 253-285.

Farley, C. T., Glasheen, J. and McMahon, T. A. (1993). "Running springs: speed and animal size." *J. Exp. Biol.* 185, 71-86.

Lee, C. R. and Farley, C. T. (1998). "Determinants of the center of mass trajectory in human walking and running." *J. Exp. Biol.* 201,2935-2944.

Lacker. H. M., Choi TH, Schenk S, Gupta B, Narcessian RP, Sisto SA, Massood S, Redling J, Engler P, Ghobadi F, McInerney VK (1997). "A mathematical model of human gait dynamics."

McMahon, T. A. and Cheng, G. C. (1990). "The mechanics of running: how does stiffness couple with speed?" *J. Biomech.* 23, (Suppl. 1), 65-78.

Cavagna, G. A., Willems, P. A., Legamandi, M. A. & Heglund, N. C. 2002. "Pendular energy transduction within the step in human walking." *J. Exp. Biol.* 205, 3413-3422.

Moretto, P., Pelayo, P. and Lafortune, M. A. (1996). "The use of Froude's numbers to normalize human gait", *In IXth Biennial Conference of the Canadian Society of Biomechanics* (ed. J. A. Hoffer, A. Chapman, J. J. Eng, A. Hodgson, T. E. Milner and D. Sanderson), pp. 274–275. Burnaby BC: Simon Fraser University.

Alexander, R. M. (1984b). "Walking and running." *American Scientist* 72, 348-354.

Alexander, R. M. "Simple models of human motion", *Applied Mechanics Review*, 48:461{469, 1995}.

Lacker, H. M., Roman, W., Narcessian, R. P., McInerne, V. K., Ghobadi. F. "The Boundary Method: A New Approach to Motor Skill Acquisition Through Model Based Drill Design." *To be Published*

Press WH, Teulolsky SA, Vetterling WT, Flannery BP. 1992. Numerical Recipes in C: The Art of Scientific Computing. 2nd ed. New York: Cambridge University Press, 714-722.

Choi, Tae Hoe. 1997. "Development of a Mathematical Model of Gait Dynamics." Dissertation New Jersey Institute of Technology, 78-91.

Schenk, S. 2000. "A Mathematical Model of Wheel Chair Racing." Dissertation New Jersey Institute of Technology,117-135.

Hatze, H. 1981. "A Comprehensive Model for Human Motion Simulation and its Application to the Take-Off Phase of the Long Jump." *J. Biomech.* 10:135-142.

Lanczos, Cornelius. 1970. The Variational Principles of Mechanics. 4th ed. New York: Dover Publications, Inc., 140-145.

McCauley, Joseph L.1997. *Classical Mechanics: Transformations, Flows, Integrable and Chaotic Dynamics*. New York: Cambridge University Press, 60-63.